MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

WEEKLY QUIZ 11 SOLUTIONS

Quiz Date: Friday, April 29, 2005

FAMILY (LAST) NAME

GIVEN (FIRST) NAME

STUDENT ID NUMBER

Please check (✔) your class

	L01	MTW 10:00	Walter Lewin
	L02	MTW 11:00	Walter Lewin
	L03	MTW 2:00	Min Chen
	L04	MTW 3:00	Min Chen

INSTRUCTIONS:

1. The FORMULA SHEET is in the back of this exam. You may tear it off. There is also an extra BLANK PAGE in case you need it.
2. This is a closed book exam. CALCULATORS, BOOKS, and NOTES are NOT ALLOWED.
3. Unless otherwise stated, to earn full credit you must show a valid DERIVATION and/or EXPLANATION of your answer, and you must express it in terms of the GIVEN VARIABLES.

ANNOUNCEMENT MADE AT THE QUIZ

Problem 1(e): Answers (i) and (iii) are identical. Circling one is equivalent to circling the other.

Problem	Maximum	Score	Grader
1	20		
2	20		
3	30		
4	30		
TOTAL	100		

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Problem 1: Basic concepts about gravity and orbits (*20 points*)

(a) Two planets, called Uno and Duo, are in circular orbits about the same star. The orbit of Duo, however, has twice the radius of the orbit of Uno. The ratio of the period T_D of Duo's orbit to the period T_U of Uno's orbit is given by $T_D/T_U =$

(i) $1/2$ (ii) $1/\sqrt{2}$ (iii) 1 (iv) $\sqrt{2}$ (v) 2 **[(vi) $2\sqrt{2}$]** (vii) 4

Explanation: Kepler's third law says the period of an orbit is proportional to the 3/2 power of the major axis length of the orbit, which for circular orbits is proportional to the 3/2 power of the radius. This can easily be verified by writing the radial component of the $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ equation. Since the orbit of Duo has twice the radius of Uno, the period is longer by a factor of $2^{3/2} = 2\sqrt{2}$.

(b) (*4 points*) Two completely different planets, which happen to also be called Uno and Duo, are each composed of the same material, which we will assume has a fixed density. The planet Duo has a radius twice as large as that of Uno. The ratio of the gravitational acceleration g_D at the surface of Duo to the gravitational acceleration g_U at the surface of Uno is given by $g_D/g_U =$

(i) $1/2$ (ii) $1/\sqrt{2}$ (iii) 1 (iv) $\sqrt{2}$ **[(v) 2]** (vi) $2\sqrt{2}$ (vii) 4

Explanation: Since the densities of the two planets are equal, the mass of Duo is 8 times the mass of Uno, since its radius is twice as large and the volume is proportional to r^3. The gravitational acceleration is given by $g = GM/r^2$. Since Duo has a mass M which is 8 times larger than Uno, and a radius r that is twice as large, the acceleration g is twice as large.

(c) (*4 points*) An astronaut of mass m is a member of the crew of a space shuttle orbiting the earth at an altitude h. If the radius of the earth is R and g is the acceleration due to gravity at the surface of the earth, the magnitude of the gravitational force on the astronaut is:

(i) mg (ii) $mg\,\frac{R}{R+h}$ (iii) $mg\,\frac{(R+h)}{R}$ **[(iv) $mg\,\frac{R^2}{(R+h)^2}$]** (v) $mg\,\frac{(R+h)^2}{R^2}$

Explanation: The gravitational force on the astronaut is $mg(h)$, where $g(h)$ is the gravitational acceleration at the altitude h. The gravitational acceleration is inversely proportional to the distance from the center of the Earth, so

$$g(h) = \frac{\alpha}{(R+h)^2}\ ,$$

$$g(0) = g = \frac{\alpha}{R^2}\ .$$

with the same constant of proportionality α. Putting these together,

$$g(h) = g\frac{R^2}{(R+h)^2}\ .$$

(d) (*4 points*) Two spherical planets, each of uniform density and total mass M, are separated from each other by a distance R. If they were point masses, Newton's law of gravity tells us that the force that each would exert on the other would have magnitude GM^2/R^2. They are not point masses, however, but instead each has a radius a, which is not negligible compared to R. When this nonzero size is taken into account the magnitude of the force between the planets is (choose one)

(i) larger than GM^2/R^2 when the spheres almost touch, but approaches GM^2/R^2 when $R \gg a$;

(ii) smaller than GM^2/R^2 when the spheres almost touch, but approaches GM^2/R^2 when $R \gg a$;

$\boxed{\text{(iii)}}$ equal to GM^2/R^2 as long as the spheres are not touching.

Explanation: It was a very famous result of Isaac Newton's that the gravitational interaction between two spherical distributions of matter is the same as if all the mass of each sphere were concentrated at its center. In the textbook this fact is proved in Section 12.6, which you were not required to read, but the fact was stated and used starting in Section 12.1. One consequence is that, to the extent that the Earth is approximated by a sphere, the force on a person of mass m *standing on the surface of the Earth has magnitude* GM_Em/R_E^2*, where* M_E *and* R_E *are the mass and radius of the Earth. This is the same as if the mass of the Earth were a point mass located at the center of the Earth, and it is much simpler than one might have guessed. After all, the Earth certainly does not look to us like a point mass 6,400 km away.*

(e) (*4 points*) A space probe measures the gravitational acceleration caused by a spherical astronomical object to be g_0, at a distance R_0 from its center. If the object is a black hole, what would be its Schwarzschild radius R_S?

$\boxed{\text{(i) } \dfrac{2R_0^2 g_0}{c^2}}$ (ii) $\dfrac{2R_0^3 g_0^2}{c^4}$ $\boxed{\text{(iii) } \dfrac{2R_0^2 g_0}{c^2}}$ (iv) $\dfrac{2g_0}{R_0^2 c^2}$ (v) $2R_0\sqrt{\dfrac{R_0 g_0}{c^2}}$

Explanation: The Schwarzschild radius of an object of mass M *is given by* $R_S = 2GM/c^2$*, but in this case we are not given the mass. But we are given the gravitational acceleration at a distance* R_0*, from which we can calculate the mass. If an object of mass* m *is introduced at a distance* R_0 *from the black hole, then the gravitational force on it will have magnitude*

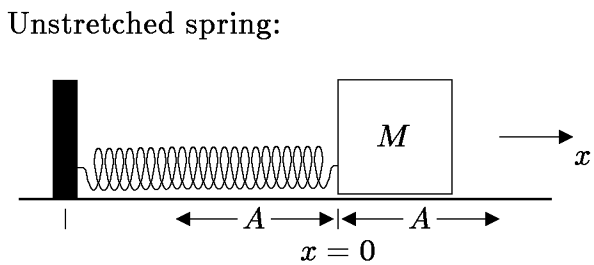
$$F = \frac{GMm}{R_0^2} ,$$

so $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$ *implies that its acceleration will have magnitude*

$$g = \frac{F}{m} = \frac{GM}{R_0^2} .$$

Solving this equation for GM *and inserting that result into the expression for the Schwarzschild radius, one finds the boxed result.*

Problem 2: Basic concepts about periodic motion (*20 points*)



(a) (*4 points*) A block attached to a spring moves along the x axis on a frictionless horizontal table, executing simple harmonic motion, as shown above. The oscillations are centered at $x = 0$, and have an amplitude A. As the block crosses $x = 0$, it is observed to have speed v_0. What is the period T of the oscillation?

(i) $\frac{v_0}{A}$ (ii) $2\pi\frac{v_0}{A}$ (iii) $\frac{A^2}{v_0}$ (iv) $2\pi\frac{A^2}{v_0}$ (v) $\frac{A}{v_0}$ (vi) $\frac{2A}{v_0}$ (vii) $\frac{\pi A}{v_0}$ **[(viii) $\frac{2\pi A}{v_0}$]**

Explanation: Simple harmonic motion can be described in general by

$$x = A\cos(\omega t + \phi) ,$$

as given on the formula sheet. The speed is then

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -A\omega\sin(\omega t + \phi) .$$

When the block crosses the origin the cosine must vanish, which means that the sine has magnitude 1. Thus $v_0 = A\omega$. The period is related to ω by $\omega = 2\pi/T$ (again on the formula sheet), so $T = 2\pi A/v_0$.

(b) (*4 points*) Now consider two blocks which, like the block in part (a), are each attached to a spring and moving along the x axis on a frictionless horizontal table, executing simple harmonic motion. They both oscillate with the same amplitude A, but the motion of the 2nd block has an angular frequency ω_2 which is twice as large as the angular frequency ω_1 for the first block. Let v_1 denote the speed of the first block when it crosses $x = A/2$, and let v_2 denote the speed of the second block when it crosses $x = A/2$. What is the ratio v_2/v_1?

(i) $1/2$ (ii) $1/\sqrt{2}$ (iii) 1 (iv) $\sqrt{2}$ **[(v) 2]** (vi) $2\sqrt{2}$ (vii) 4

Explanation: Again, the position and velocity are given generically by

$$x = A\cos(\omega t + \phi) \ ,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -A\omega\sin(\omega t + \phi) \ .$$

This time we do not have $x = 0$, but instead $x = A/2$, but the answer is actually valid for all x. For any x one has

$$|\sin(\omega t + \phi)| = \sqrt{1 - \cos^2(\omega t + \phi)} = \sqrt{1 - (x/A)^2} \ ,$$

so

$$\left|\frac{\mathrm{d}x}{\mathrm{d}t}\right| = A\omega\sqrt{1 - (x/A)^2} \ .$$

So v is proportional to ω, and the block with twice the value of ω has twice the value of v.

(c) (*4 points*) For the same two blocks described in part (b), let a_1 denote the acceleration of the first block when it crosses $x = A/2$, and let a_2 denote the acceleration of the second block when it crosses $x = A/2$. What is the ratio a_2/a_1?

(i) $1/2$ (ii) $1/\sqrt{2}$ (iii) 1 (iv) $\sqrt{2}$ (v) 2 (vi) $2\sqrt{2}$ **[(vii) 4]**

Explanation: Following the same analysis as in part (b), the acceleration is given by

$$\frac{\mathrm{d}^2x}{dt^2} = -A\omega^2\cos(\omega t + \phi) \ ,$$

so

$$\frac{\mathrm{d}^2x}{dt^2} = -\omega^2 x \ ,$$

where we used the original equation $x = A\cos(\omega t + \phi)$. So a is proportional to ω^2, and the block with twice the value of ω has 4 times the value of a.

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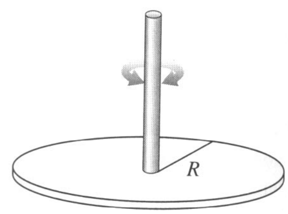
(d) (*4 points*) A pendulum is described by an angle θ, which measures its deviation from the vertical. It undergoes simple harmonic motion with frequency f and amplitude θ_m. At $t = 0$ the pendulum is at its maximum angle, $\theta = \theta_m$. Which of the following functions $\theta(t)$ describes the motion of the pendulum?

(i) $\theta_m \sin(ft)$ (ii) $\theta_m \sin(2\pi ft)$ (iii) $\theta_m \cos(ft)$ $\boxed{\text{(iv) } \theta_m \cos(2\pi ft)}$

(v) $\frac{1}{\sqrt{2}}\theta_m \sin(ft)$ (vi) $\frac{1}{\sqrt{2}}\theta_m \sin(2\pi ft)$ (vii) $\frac{1}{\sqrt{2}}\theta_m \cos(ft)$ (viii) $\frac{1}{\sqrt{2}}\theta_m \cos(2\pi ft)$

Explanation: The amplitude is the maximum deviation from the central value. Both the sine and cosine functions oscillate about a central value of zero, with amplitude 1, so the functions (i)-(iv) have the specified amplitude θ_m, while the functions (v)-(viii) are eliminated because they have amplitude $\theta_m/\sqrt{2}$. Functions (ii) and (iv) both have the specified frequency f, because the argument of the sine and cosine function is specified in radians. Since 2π radians corresponds to one period of the sine or cosine function, both functions (ii) and (iv) will vary through $f\Delta t$ periods in a time Δt, so f is the number of periods per unit time, which is the frequency. Finally, we are told that the function is at its maximum when $t = 0$, but function (ii) vanishes when $t = 0$. Function (iv) is therefore the only function that satisfies all the stated properties.

(e) (*4 points*) A horizontal circular disk of radius R and mass M has uniform density, and is suspended from a thread to create a torsional pendulum. When the thread twists through an angle θ, it creates a restoring torque $\boldsymbol{\tau} = -\kappa\theta$. What is the frequency f of this pendulum?



(i) $\dfrac{R}{2\pi}\sqrt{\dfrac{M}{2\kappa}}$ (ii) $2\pi R\sqrt{\dfrac{M}{2\kappa}}$ (iii) $\dfrac{R}{2\pi}\sqrt{\dfrac{M}{\kappa}}$ (iv) $2\pi R\sqrt{\dfrac{M}{\kappa}}$

(v) $\boxed{\dfrac{1}{2\pi R}\sqrt{\dfrac{2\kappa}{M}}}$ (vi) $\dfrac{2\pi}{R}\sqrt{\dfrac{2\kappa}{M}}$ (vii) $\dfrac{1}{2\pi R}\sqrt{\dfrac{\kappa}{M}}$ (viii) $\dfrac{2\pi}{R}\sqrt{\dfrac{\kappa}{M}}$

Explanation: The equation of motion for the torsional pendulum is

$$\boldsymbol{\tau} = -\kappa\theta = I\alpha = I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} \ ,$$

where $I = \frac{1}{2}MR^2$ *for a uniform disk. So*

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\frac{2\kappa}{MR^2}\theta \ .$$

This can be written as

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -\omega^2\,\theta \ ,$$

where $\omega = \sqrt{2\kappa/(MR^2)}$. *This equation has the standard simple harmonic motion solution*

$$\theta = A\cos(\omega t) \ .$$

We are asked for the frequency f, *given by*

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi R}\sqrt{\frac{2\kappa}{M}} \ .$$

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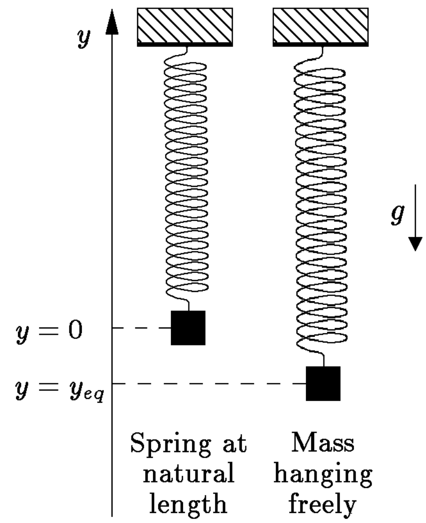
Problem 3: A mass suspended on a spring (*30 points*)

A block of mass M is suspended from a massless ideal spring of spring constant k. The coordinate system is defined so that y is directed vertically upwards and $y = 0$ when the spring is at its natural (i.e., unstretched) length. The mass is first positioned at $y = 0$, and then lowered gently until it is hanging freely from the spring.



(a) (*3 points*) What is the value of y_{eq}, the coordinate of the equilibrium point at which the block comes to rest?

For the remaining parts you may treat y_{eq} as if it were a given variable, whether or not you answered (a) correctly.

(b) (*3 points*) When the block comes to rest, what is its gravitational potential energy U_g, compared to its value when the block was at $y = 0$?

(c) (*3 points*) When the block comes to rest, what is the potential energy U_s stored in the spring, compared to its value when the block was at $y = 0$?

(d) As the block was lowered from $y = 0$ to its equilibrium position,

(i) (*2 points*) What was the total work done on the block?

(ii) (*2 points*) What was the work done by gravity?

(iii) (*2 points*) What was the work done by the spring force?

(iv) (*2 points*) Was there any other agent that did work on the block? If so, what was that agent, and how much work did it do?

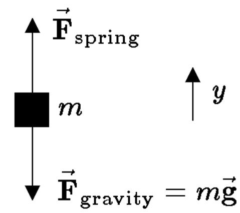
Now suppose that instead of being lowered gently, the mass is simply released from rest at $y = 0$ and allowed to fall under the combined effects of gravity and the spring force.

(e) (*3 points*) At what speed is it moving when it crosses $y = y_{eq}$?

(f) (*5 points*) What is the period T of the oscillations of the mass?

(g) (*5 points*) What is the amplitude of these oscillations? That is, how far up or down will the mass move from its average position?

(a) When the mass comes rest there is no net force acting on it, so



$$\vec{\mathbf{F}}_{\text{spring}} + M\vec{\mathbf{g}} = 0 \ .$$

Using Hooke's law, the y-component of this equation becomes:

$$F_y = -ky_{\text{eq}} - mg = 0 \ ,$$

so

$$y_{eq} = \boxed{-\frac{mg}{k}}$$

(b) The gravitational potential energy of the block is given by

$$U_g = Mgy + \text{constant} \ ,$$

so the potential energy at y_{eq} relative to $y = 0$ is given by

$$U_g(y_{eq}) - U_g(0) = \boxed{mgy_{eq}} = \boxed{-\frac{(Mg)^2}{k} \ .}$$

(c) For the potential energy stored in the spring,

$$U_s = \frac{1}{2}k(\text{extension of spring})^2 = \frac{1}{2}ky^2 \ .$$

The potential energy at y_{eq} compared to its value at $y = 0$ is then given by

$$U_s(y_{eq}) - U_s(0) = \boxed{\frac{1}{2}ky_{eq}^2} = \boxed{\frac{1}{2}\frac{(Mg)^2}{k} \ .}$$

(d) (i) By the work-energy theorem, the total work done on the block is the change in its kinetic energy. But it starts at rest and ends at rest, so its initial and final kinetic energy are zero, so the change in kinetic energy is zero. The total work done on the block is therefore $\boxed{\text{zero.}}$

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(ii) The work done by gravity is the negative of the change in potential energy calculated in part (b):

$$W_g = \boxed{-mgy_{eq}} = \boxed{\frac{(Mg)^2}{k}\ .}$$

To understand the sign, remember that the lowering of an object always decreases the gravitational potential energy, but gravity is pushing downward on the object, so it does positive work.

(iii) As above, the work done by the force is the negative of the change in potential energy that we already calculated.

$$W_s = \boxed{-\frac{1}{2}ky_{eq}^2} = \boxed{-\frac{1}{2}\frac{(Mg)^2}{k}\ .}$$

(iv) The work done by gravity plus the work done by the spring do not add up to the total work done on the block, so some other agent must have been supplying negative energy. The extra object was the hand, which during the lowering process must exert an upward force to prevent the block from picking up speed. Since the total work done is zero (see (i) above), the work done by the hand is

$$W_{\text{hand}} = -W_g - W_s = \boxed{Mgy_{eq} + \frac{1}{2}ky_{eq}^2} = \boxed{-\frac{(Mg)^2}{k}}\ .$$

(e) To answer this question, we can use the conservation of mechanical energy. Defining the zero of potential energy for both gravity and the spring to be at $y = 0$, the initial energy E_{initial} of the block is zero. At the final time, when the block crosses $y = y_{eq}$, the total mechanical energy is

$$E_{\text{final}} = Mgy_{eq} + \frac{1}{2}ky_{eq}^2 + \frac{1}{2}Mv^2\ .$$

Since this energy must equal $E_{\text{initial}} = 0$, we have

$$v = \boxed{\sqrt{-2gy_{eq} - \frac{k}{M}y_{eq}^2}\ .}$$

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If one substitutes the value for y_{eq}, which was not required for the quiz, one finds

$$v = \boxed{\; g\sqrt{\frac{M}{k}} \; .}$$

(f) The subsequent motion can be found most clearly by defining a new variable, $y' = y - y_{eq}$, so $y' = 0$ at the equilibrium position. Then if the mass is displaced from $y' = 0$ there is a net restoring force $F_y(y') = -ky'$ in the y' (or y) direction. Thus the equation of motion for the mass M becomes

$$M\frac{\mathrm{d}^2 y'}{\mathrm{d}t^2} = -ky' \; ,$$

or

$$\frac{\mathrm{d}^2 y'}{\mathrm{d}t^2} = -\frac{k}{M}\, y' = -\omega^2 y' \; , \qquad \text{with } \omega = \sqrt{\frac{k}{M}} \; .$$

This is exactly the differential equation for simple harmonic motion with angular frequency ω. The period T is then given y

$$T = \frac{2\pi}{\omega} = \boxed{\; 2\pi\sqrt{\frac{M}{k}} \; .}$$

(g) Simple harmonic motion is always symmetric on both sides of the equilibrium position, which here is $y = y_{eq}$. The maximum height is clearly the starting height, $y = 0$. We know the block has enough energy to reach this point, since it started there. It cannot get higher, because we know that it reaches the point with zero kinetic energy. If the block were to go higher both its gravitational and spring potential energy would increase, and since there is no way that the kinetic energy could decrease to less than zero, the block simply cannot go higher. Thus the distance between its maximum height and its equilibrium position is $|y_{eq}|$, which is the amplitude:

$$\boxed{A = |y_{eq}| = \left|\frac{Mg}{k}\right| \; .}$$

Name

Problem 4: Space Shuttle Maneuvers (*30 points*)

A space shuttle of mass M_S travels in a circular orbit, of radius R_0, about the Earth. Let M_E denote the mass of the Earth, and let G denote Newton's constant. Treat the Earth as if it were at rest in an inertial reference frame, and assume that the shuttle is high enough so that atmospheric drag can be ignored.

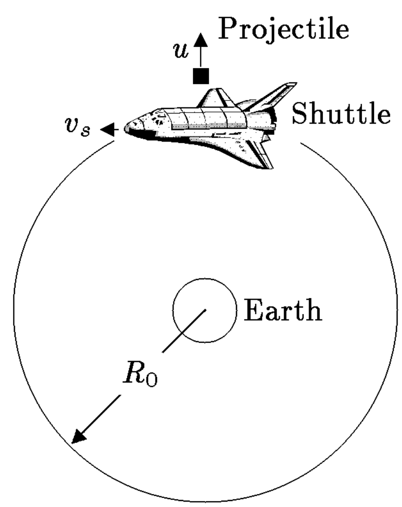
(a) (*5 points*) What is the magnitude a of the acceleration of the shuttle? Express your answer in terms of some or all of the variables G, M_E, M_S, and R_0. What is the direction of the acceleration?

(b) (*4 points*) What is the speed v_S of the shuttle in its orbit? Express your answer **ONLY** in terms of the magnitude of the acceleration a and the radius R_0 of the orbit.

In the following questions, you may express your answer to each part in terms of the given variables and the symbols representing the answers to any previous parts, regardless of whether the previous parts were answered correctly.

(c) (*5 points*) What is the magnitude L_S of the angular momentum of the shuttle about the center of the Earth?

Suppose that the shuttle fires a projectile of mass m in a direction radially outward from the Earth, with a speed u. That is, the projectile is given a velocity, **relative to the satellite,** which has magnitude u and is directed opposite to the direction of the Earth. Assume that the projectile is light enough (i.e., $m \ll M_S$) so that the recoil of the shuttle can be ignored.



(d) (*5 points*) Immediately after the projectile is ejected, what is its total mechanical energy? Define the gravitational potential energy so that it would vanish for an object infinitely far from the Earth.

(e) (*3 points*) Immediately after the projectile is ejected, what is the magnitude L_m of its angular momentum about the center of the Earth?

(f) (*8 points*) Let R_{max} denote the maximum distance from the center of the Earth that the projectile will reach, and let v_{max} denote the speed of the projectile (relative to the inertial frame of the Earth) when it reaches the maximum distance. (Note that v_{max} is not the maximum velocity.) Write down two equations which can be solved to find R_{max} and v_{max}. Do not solve the equations.

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(a) The centripetal acceleration is due to the gravitational force GM_EM_S/R_0^2, so the radial component of $\vec{\mathbf{F}} = M_S\vec{\mathbf{a}}_S$ implies

$$\boxed{a = \frac{GM_E}{R_0^2}\ .}$$

(b) The acceleration of an object moving in a uniform circle is radially inward, with magnitude given by $a = v^2/R$, so in this case

$$\boxed{v_S = \sqrt{aR_0}\ .}$$

You were not asked to substitute the expression above for a, but if you did you would find

$$v_S = \sqrt{\frac{GM_E}{R_0}}\ .$$

(c) The magnitude of the angular momentum is given by

$$L_S = \left|\vec{\mathbf{r}} \times \vec{\mathbf{p}}\right| = \boxed{M_S v_S R_0\ .}$$

(d) The radial velocity of the projectile will be u, and the tangential velocity will be the same as the shuttle, v_S. Since these are perpendicular, the kinetic energy is

$$E_k = \frac{1}{2}m\left(v_S^2 + u^2\right)\ .$$

The gravitational potential energy is $-GM_Em/R_0$, so

$$\boxed{E_{\text{mech}} = \frac{1}{2}m\left(v_S^2 + u^2\right) - \frac{GM_Em}{R_0}\ .}$$

(e) The angular momentum is found by the same method as in (c). The radial component of the velocity does not contribute to $\vec{\mathbf{r}} \times \vec{\mathbf{p}}$, so

$$L_S = \left|\vec{\mathbf{r}} \times \vec{\mathbf{p}}\right| = \boxed{m v_S R_0\ .}$$

(f) One equation describes the conservation of angular momentum:

$$\boxed{m v_S R_0 = m v_{\max} R_{\max} \ ,}$$

where we used the fact that at the maximum altitude, the velocity $v_{\max}$ of the projectile must be purely tangential. The other is the conservation of mechanical energy:

$$\boxed{\frac{1}{2} m \left(v_S^2 + u^2\right) - \frac{G M_E m}{R_0} = \frac{1}{2} m v_{\max}^2 - \frac{G M_E m}{R_{\max}} \ .}$$

You were not asked to solve these equations, but if you did you would find

$$v_{\max} = v_S - u$$

$$R_{\max} = \frac{v_S}{v_S - u} R_0 \ .$$

One solves a quadratic equation which has two roots, but one can see that one root represents the minimum radius of the orbit, and the root described above is the maximum radius.