MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

WEEKLY QUIZ 12 SOLUTIONS

Quiz Date: Friday, May 6, 2005

FAMILY (LAST) NAME

GIVEN (FIRST) NAME

STUDENT ID NUMBER

Please check (✔) your class ☞

	L01	MTW 10:00	Walter Lewin
	L02	MTW 11:00	Walter Lewin
	L03	MTW 2:00	Min Chen
	L04	MTW 3:00	Min Chen

INSTRUCTIONS:

1. The FORMULA SHEET is in the back of this exam. You may tear it off. There is also an extra BLANK PAGE in case you need it.
2. This is a closed book exam. CALCULATORS, BOOKS, and NOTES are NOT ALLOWED.
3. Unless otherwise stated, to earn full credit you must show a valid DERIVATION and/or EXPLANATION of your answer, and you must express it in terms of the GIVEN VARIABLES.

Problem	**Maximum**	**Score**	**Grader**
1	25		
2	35		
3	40		
TOTAL	100		

Name ______________________

Problem 1: Basic concepts about special relativity (*25 points*)

(a) (*5 points*) A rocket flies at a relativistic speed v in a straight line from space station A to space station B, which are at rest relative to each other. The space stations keep time on a system of clocks which are synchronized in their common rest frame. Measured on the clock inside the rocket, the elapsed time for the trip is Δt. If t_A and t_B denote the time of the take-off and landing, respectively, on the space station clocks, what is $t_B - t_A$? Let γ denote $1/\sqrt{1 - v^2/c^2}$.

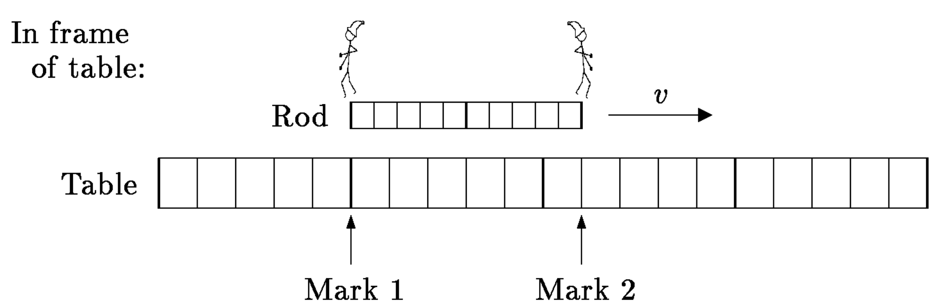
(i) Δt | **[(ii) $\gamma \Delta t$]** | (iii) $\Delta t/\gamma$ | (iv) $\gamma^2 \Delta t$ | (v) $\Delta t/\gamma^2$

Explanation: In the common reference frame of the two space stations, the clock aboard the rocket is moving, and hence appears to run slowly by a factor of γ, where γ is always greater than 1 (as long as v is nonzero). Hence the time interval measured by the space stations, $t_B - t_A$, is longer by a factor of γ than the time interval Δt measured on the rocket clock.

(b) (*5 points*) A measuring rod with length ℓ_0 in its own rest frame is pulled across the surface of a horizontal table at relativistic speed v, along the direction of its length. At a fixed time in the rest frame of the table, two elves make scratch marks in the table at the locations of the two endpoints of the rod. When measured in the rest frame of the table, how far apart are these marks? Again let γ denote $1/\sqrt{1 - v^2/c^2}$.

(i) ℓ_0 | (ii) $\gamma \ell_0$ | **[(iii) ℓ_0/γ]** | (iv) $\gamma^2 \ell_0$ | (v) ℓ_0/γ^2

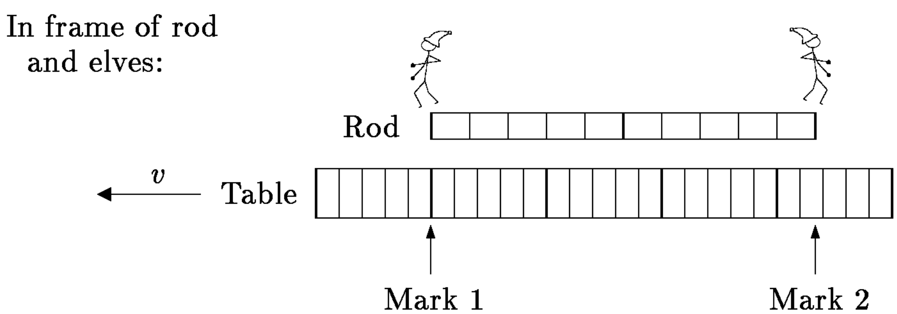
Explanation: Since the two scratch marks were made simultaneously in the rest frame of the table, the distance between them is a measurement of the length of the measuring rod as seen in the frame of reference of the table, as shown below. The diagram shows schematically the rod, the elves, and a tape measure that one can imagine being stretched out across the table, at rest with respect to the table. Since the rod is moving in the direction of its length, it will appear Lorentz-contracted. The markings shown both on the rod and on the measuring tape are all spaced 10 cm apart in the rest frame of the object, but the markings on the rod appear more closely spaced because the rod is moving relative to the frame of the diagram. Thus the scratch marks will be closer together than the rest length ℓ_0 of the rod by a factor of γ.



(c) (*5 points*) Consider the same measuring rod as described in (b), being pulled across a horizontal table at relativistic speed v. This time imagine elves that are riding on the rod, and have their elvish wristwatches synchronized in the frame of the rod. At the same time according to these wristwatches, the elves carve marks in the table at both ends of the rod. In this case, when measured in the rest frame of the table, how far away are the marks? As usual, let γ denote $1/\sqrt{1 - v^2/c^2}$.

(i) ℓ_0 | **(ii) $\gamma\ell_0$** | (iii) ℓ_0/γ | (iv) $\gamma^2\ell_0$ | (v) ℓ_0/γ^2

Explanation: Since the elves' watches are synchronized in the frame of the rod, the distance between the scratch marks as seen by the elves is a measure of length in the reference frame of the rod, as shown on the diagram below. That is, from the point of view of the elves, they are using their measuring rod to make scratch marks on the table that are a distance ℓ_0 apart. But the table is moving in their frame of reference, so it will be Lorentz-contracted by a factor of γ. Thus, the distance between the two marks, as measured on a tape measure at rest with respect to the table, is larger than ℓ_0 by a factor of γ.



(d) (*5 points*) A very high speed train moves along a straight track from your right to your left. The train has length ℓ_0 in its own rest frame. There are clocks at the front and rear of the train, which are synchronized with each other in the rest frame of the train. At a fixed time in your reference frame, observers who happen to be situated at the two ends of the train note the time readings on the two clocks, calling them t_{left} and t_{right}, referring to the left and right ends of the train, respectively, as seen by you. What is $\Delta t \equiv t_{\text{left}} - t_{\text{right}}$?

(i) $\dfrac{\gamma v \ell_0}{c^2}$ (ii) $\dfrac{v \ell_0}{c^2}$ (iii) $\dfrac{v \ell_0}{\gamma c^2}$ (iv) $-\dfrac{\gamma v \ell_0}{c^2}$ **[(v) $-\dfrac{v \ell_0}{c^2}$]** (vi) $-\dfrac{v \ell_0}{\gamma c^2}$

(vii) $\dfrac{\gamma v^2 \ell_0}{c^2}$ (viii) $\dfrac{v^2 \ell_0}{c^2}$ (ix) $\dfrac{v^2 \ell_0}{\gamma c^2}$ (x) $-\dfrac{\gamma v^2 \ell_0}{c^2}$ (xi) $-\dfrac{v^2 \ell_0}{c^2}$ (xii) $-\dfrac{v^2 \ell_0}{\gamma c^2}$

Explanation: We have two clocks, one at each end of the train, which are synchronized in the frame of the train. The train is moving relative to you, however, so that means that the clocks will not appear simultaneous to you. As it says on the formula sheet under Relativity of Simultaneity, "the trailing clock will appear to read a time which is later than the leading clock by an amount $\beta \ell_0 / c$," where $\beta = v/c$. Since the train is moving from right to left, the right end is the trailing ending, so t_{right} should be larger than t_{left} by $v\ell_0/c^2$. We are asked for $t_{\text{left}} - t_{\text{right}}$, which is then $-v\ell_0/c^2$. Although there is not normally partial credit for multiple choice problems, in this case we gave 2 out of 5 points to students who chose (ii), which differs from the right answer only by a sign. While sign errors are just as likely to cause bridges to fall down as other kinds of errors, they are more likely than other errors to occur as a result of minor carelessness.

(e) (*5 points*) Two twins, Karla and Marla, are born in the year 2300. In 2318, Karla takes off on a spaceship which travels at a relativistic speed in a straight line for many years, and then stops, reverses its direction, and returns to Earth in the year 2360. When the 60-year-old Marla looks at her twin sister Karla, she is immediately jealous that her sister looks (and really is) 20 years younger than she is, due to relativistic time dilation. However, just as Karla has been moving at high speed relative to Marla, Marla has been moving at high speed relative to Karla. So what does Karla see when she looks at her sister Marla? Choose the best answer.

(i) To Karla it looks like Marla is younger, because in her own reference frame Karla has been at rest, and Marla has been moving at high speed.

(ii) To Karla it looks like they are the same age, because the time dilation effect is only apparent, not real.

(iii) Karla agrees with Marla that Marla has aged 20 years more than Karla. This asymmetry can be explained because it was really Karla who was in motion, while Marla was always at rest.

(iv) Karla agrees with Marla that Marla has aged 20 years more than Karla. This asymmetry can be explained because Karla has undergone dramatic acceleration when she began her space journey, when the spaceship reversed its direction, and when it came to rest on Earth. Marla, on the other hand, has undergone no such acceleration.

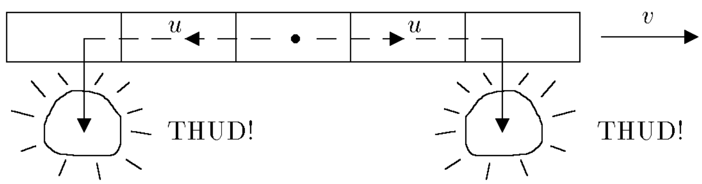
(v) Karla agrees with Marla that Marla has aged 20 years more than Karla. This asymmetry can be explained because Marla spent her whole life on Earth, while Karla was in deep space where relativistic effects can occur.

Explanation: This problem is the famous "twin paradox," and if you read the section in Young & Freedman on the twin paradox you would certainly have had no trouble with it. Apparently many of you did not read it. Even so, if you understand the basics of special relativity, it should be easy to exclude the wrong answers. Concerning choice (i), it's important to keep in mind that although relativity implies that many observations yield different results for different observers, it does not go so far as to say that each observer lives in her own imaginary world which has no connection to the world of other observers. Using the word "event" to mean an occurrence that happens at a definite place and time, then all observers see the same events. It's just that two observers will not always agree on the space and time coordinates of an event. However, if two events coincide for one observer (i.e., happen at the same place and at the same time), then they will coincide for ***all*** *observers. So if Karla's body reaches the biological age of 40 years at the same place and time that Marla's body reaches the biological age of 60 years, all observers will agree that those two events coincided. Thus (i) is excluded. Choice (ii), which proposes that time dilation is not real, does not really make any sense. Time dilation is real, and so for example unstable particles traveling near the speed of light really do decay more slowly, because their internal clock is slowed by time dilation. Choice (iii), which proposes that the relative aging depends on which sister was in motion, contradicts the basic hypothesis of relativity, which is that "The laws of physics are the same in every inertial frame of reference." All inertial frames are equivalent, so there is no meaning to saying that one sister was at rest while the other was not. Finally, choice (v) suggests that relativity somehow only works in deep space. On the contrary, since the time of Newton our understanding of physics has been based on the assumption that the laws of physics that we measure on Earth hold throughout the universe. While this assumption cannot really be proven, it has never been found to fail. That leaves (iv), which is the right answer. Karla's and Marla's histories are not equivalent, because one has experienced significant acceleration, and the other has not. Relativity says that all inertial frames of reference are equivalent, but accelerating frames of reference are not.*

Name ______________________________

Problem 2: Dynamic Duo Jumps Off a Fast Train (*35 points*)

Batman and Robin are standing next to each other on the roof of a train, which is moving at a steady velocity of magnitude v. (Since this is a relativity problem, v is of course not negligible compared to c. Amtrak would have a hard time keeping up.) The superheroes simultaneously set their stopwatches to zero, and then by prearrangement they start running along the train in opposite directions at speed u, measured in the frame of reference of the train. They each run for a time Δt, as measured on their own stopwatches, and then they each jump from the train. They hit the ground with a sharp thud, and come instantly to a stop. Miraculously neither member of the Dynamic Duo is hurt. Assume that the time required for the jump is negligible, so that they land instantaneously directly below the point from which they jumped.



(a) (*5 points*) As measured in the frame of reference of the train, how far apart are Batman and Robin when they jump?

(b) (*10 points*) As measured on the ground, how far apart are the two points at which the pair hit the ground?

(c) (*10 points*) As seen from the ground, do the two superheroes hit the ground simultaneously? If not, calculate the time difference.

(d) (*10 points*) From the point of view of an observer on the ground, what is the distance between the point where Batman began to run, and the point where he jumps. Assume that Batman is the member of the pair who runs forward.

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(a) In the frame of reference of the train the stopwatches are moving, so they run slowly by a factor of $\gamma_u = 1/\sqrt{1-u^2/c^2}$. Thus each superhero runs a for a time $\gamma_u \, \Delta t$, and the distance run by each superhero is then $\Delta x = u\gamma_u \, \Delta t$. Since each superhero runs this distance, their separation when they jump is

$$\boxed{\ell_{\text{train}} = 2u\gamma_u \, \Delta t \ , \ \text{where} \ \ \gamma_u = \frac{1}{\sqrt{1-u^2/c^2}} \ .}$$

(b) We continue to analyze the problem in the frame of reference of the train. Since both stopwatches run slowly by the same factor, in this frame of reference the two superheroes jump simultaneously. To calculate the distance between the two jumps as measured on the ground, one can imagine a long tape measure stretched along the ground, and stationary with respect to the ground. In the frame of the train this tape measure will appear compressed by a factor $\gamma_v = 1/\sqrt{1-v^2/c^2}$. Thus, the points at which the superheroes jump will be separated in their readings on the tape measure by

$$\boxed{\ell_{\text{ground}} = \gamma_v \ell_{\text{train}} = 2u\gamma_v\gamma_u \, \Delta t \ , \ \text{where} \ \ \gamma_v = \frac{1}{\sqrt{1-v^2/c^2}} \ .}$$

(c) We will continue to work in the frame of reference of the train. To calculate the time-separation of the jumps as seen from the ground, we can imagine two ground-based clocks, one at the site of each jump. These clocks are synchronized in the frame of reference of the ground, and the problem asks about the difference of the readings on these two clocks at the time of the jumps. Since the jumps are simultaneous in the frame of the train, the problem is simply one of calculating the apparent lack of synchronization of the ground-based clocks as seen in the train's frame of reference. According to special relativity, the "trailing" clock will read later. Since an observer on the train would see the ground moving from the front of the train to the back, it is the clock nearer the front of the train which reads later. The time difference is given by v/c^2 times the distance between the clocks, ℓ_{train}, as measured in the frame of the clocks. ℓ_{train} is just the answer to part (b).

Thus an observer on the ground sees the superhero toward the front of the train jump ***later*** by an amount

$$\frac{2uv\gamma_v\gamma_u\Delta t}{c^2}.$$

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(d) I'll continue to describe the situation in the reference frame of the train, and again I will imagine that there is a long tape measure stretched along the ground. As Batman starts to run, he can make a mark on the tape measure immediately below him. In the reference frame of the train, this mark will move toward the back of the train at speed v. Batman runs for a time $\gamma_u\,\Delta t$, during which time he moves forward by a distance $u\gamma_u\,\Delta t$, and the mark on the tape moves backwards by $v\gamma_u\,\Delta t$. Thus, when he jumps, the separation between Batman and the mark on the tape is $(u+v)\gamma_u\,\Delta t$, as measured in the frame of the train. But in this frame the tape measure is compressed by a factor of γ_v, so the distance reading on the tape measure, which is the distance from the point of view of an observer on the ground, is given by

$$\boxed{\text{Separation}_{\text{ground}} = (u+v)\gamma_v\gamma_u\,\Delta t\ .}$$

ALTERNATIVE SOLUTION 1: Once the problem is understood in the frame of reference of the train, one could use the Lorentz transformations to transform the results to the frame of reference of the ground, to answer parts (b), (c), and (d).

I will let the ground-based coordinates be called (x,t), while the coordinates in the frame of reference of the train will be called (x',t'). The relation between the two frames is then given by the Lorentz transformation:

$$x = \gamma_v(x' + vt') \qquad (1)$$

$$t = \gamma_v\left(t' + \frac{vx'}{c^2}\right), \qquad (2)$$

where

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}\ .$$

Signs are important, so we should note that Eq. (1) implies that if an object is at rest on the train, so that x' does not change, then x increases with time. Thus our equations correspond to the case where the train is moving with speed v in the positive x direction.

In the train's frame of reference, having answered part (a), we know that the two jumps occur at $x'_1 = -u\gamma_u\Delta t$ and $x'_2 = +u\gamma_u\Delta t$. They both occur at $t' = \gamma_u\,\Delta t$. By straightforward substitution into the Lorentz transformation equations (1) and (2), one finds

$$\begin{aligned} x_1 &= \gamma_v(-u\gamma_u\,\Delta t + v\gamma_u\,\Delta t) \\ &= (v-u)\gamma_v\gamma_u\,\Delta t\ ; \\ t_1 &= \gamma_v\left[\gamma_u\,\Delta t + \frac{v}{c^2}(-u\gamma_u\,\Delta t)\right] \\ &= \left(1-\frac{vu}{c^2}\right)\gamma_v\gamma_u\,\Delta t\ ; \\ x_2 &= \gamma_v(u\gamma_u\,\Delta t + v\gamma_u\,\Delta t) \\ &= (v+u)\gamma_v\gamma_u\,\Delta t\ ; \\ t_2 &= \gamma_v\left[\gamma_u\,\Delta t + \frac{v}{c^2}(u\gamma_u\,\Delta t)\right] \\ &= \left(1+\frac{vu}{c^2}\right)\gamma_v\gamma_u\,\Delta t\ . \end{aligned}$$

Then one can immediately read off the answer to (b),

$$\boxed{x_2 - x_1 = 2u\gamma_v\gamma_u\,\Delta t\ ,}$$

and also the answer to (c):

$$\boxed{t_2 - t_1 = \frac{2vu}{c^2}\gamma_v\gamma_u\,\Delta t\ .}$$

Since this quantity is positive, it means that the jump time of the runner who goes forward (t_2) is later than the jump time of the runner who goes backwards. Finally, the total distance run by the superhero who runs forward is just

$$\boxed{x_2 = (v+u)\gamma_v\gamma_u\,\Delta t\ ,}$$

answering part (d).

ALTERNATIVE SOLUTION 2: We can bypass the frame of reference of the train, and describe everything directly in the frame of reference of the ground.

The key subtlety is that Batman's velocity in the ground frame is ***not*** $v+u$. (If velocities added so simply, then the speed of light would obviously not be invariant.) Instead we must use the formula for the relativistic addition of velocities, which implies that Batman's velocity in the ground frame is given by

$$v_B^x = \frac{u+v}{1+\frac{uv}{c^2}}\ .$$

One can compute Batman's γ-factor from this velocity, finding after some algebra that

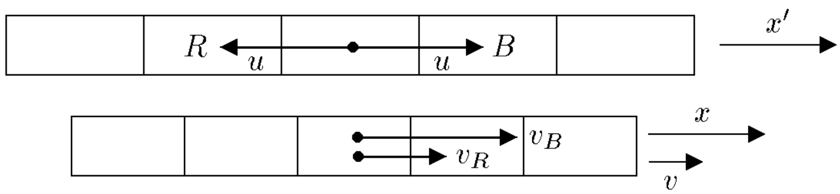
$$\gamma_B = \frac{1}{\sqrt{1 - (v_B^x/c)^2}} = \gamma_u \gamma_v \left(1 + \frac{uv}{c^2}\right) \ .$$

For Robin's velocity the technique is the same, except u enters with the opposite sign. Thus

$$v_R^x = \frac{v - u}{1 - \frac{uv}{c^2}}$$

$$\gamma_R = \gamma_u \gamma_v \left(1 - \frac{uv}{c^2}\right) \ .$$

The following diagram illustrates the train frame of reference on top, and the frame of reference of the ground on the bottom:



Note that v_R represents the component of Robin's velocity in the positive x direction. The picture is drawn as if $v > u$, in which case v_R is positive, but the equations will be correct no matter whether u or v is larger.

One can now write down the answers quite easily. In this frame Batman will run for a time $\gamma_B \Delta t$, and will therefore travel a distance $\gamma_B v_B \Delta t$ before he jumps. With a similar expression for Robin, the separation between the two superheroes when they jump is given by

$$\text{Separation} = \gamma_B v_B \Delta t - \gamma_R v_R \Delta t = 2u\gamma_u\gamma_v \Delta t \ .$$

The time difference between the two jumps is given by

$$\text{Time Difference} = (\gamma_B - \gamma_R)\, \Delta t = \frac{2\gamma_u \gamma_v uv}{c^2} \Delta t \ .$$

And the distance that Batman runs is given by

$$\gamma_B v_B \Delta t = (u + v)\gamma_v \gamma_u \, \Delta t \ .$$

Thus, the answers to (b), (c), and (d) all agree with the previous calculation.

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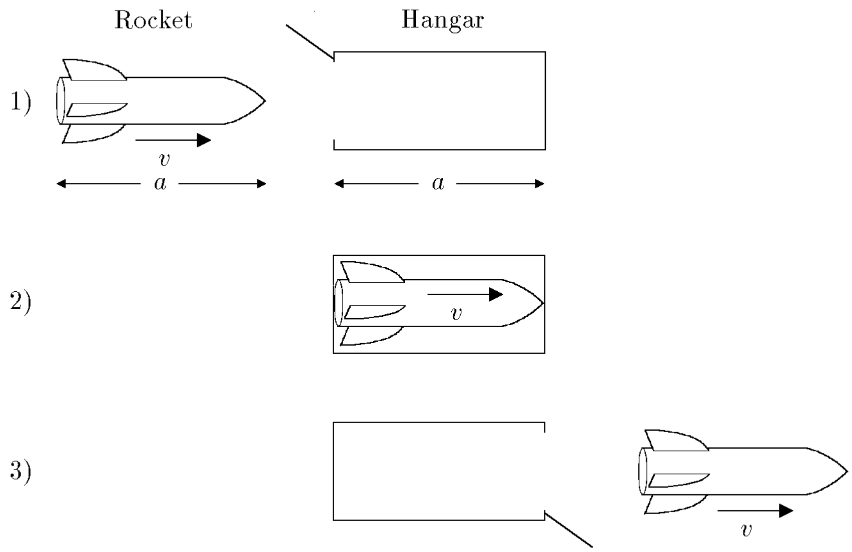
Problem 3: A Rocket Ship Squeezed in a Hangar (*40 points*)



A rocket ship moves at speed v through a hangar which is at rest, and which has a door at either end. As observed in the rest frame (*i.e.*,the reference frame of the hangar), both the rocket ship and the hangar have exactly the same length, a. Picture (1) above shows a snapshot diagram of the rocket approaching the hangar, with the left door open. All three pictures are snapshot diagrams in the rest frame of the hangar, indicating the location of all objects at a constant time in that rest frame. In picture (2) the rocket has moved completely inside the hangar; the left door has been closed, and for an instant the rocket is completely enclosed. The door on the right is immediately opened, however, just in time to prevent it from being smashed by the rocket's nose cone. Picture (3) shows the rocket well outside the hangar.

(a) (*5 points*) What is the length ℓ_R of the rocket in its own rest frame?

(b) (*5 points*) What is the length ℓ_H of the hangar in the rest frame of the rocket?

In the subsequent parts, you may treat ℓ_R and ℓ_H as given variables, whether or not you have answered (a) and (b) correctly.

(c) (*3 points*) Assume that the rocket has clocks at both ends, and that these clocks are synchronized in the reference frame of the rocket. In picture (2), when both doors are instantaneously closed and the rocket is entirely contained inside the hangar (as seen in the reference frame of the hangar), the time on the clock at the tail reads t_1, and the clock at the nose cone reads t_2. Is t_2 less than t_1, equal to t_1, or greater than t_1?

(d) (*7 points*) Give an expression for $\Delta t \equiv t_2 - t_1$ in terms of the given variables.

— *Problem 3 Continues* —

(e) (*10 points*) Draw a snapshot diagram in the frame of the ***rocket***, at the instant that the door at the tail of the rocket is closed.

(f) (*5 points*) At the instant the hangar door at the rear of the rocket is closed, a light signal is sent from the rear of the rocket toward the front. In the frame of reference of the rocket, how long does it take for the signal to reach the front of the rocket?

(g) (*5 points*) In the rest frame (*i.e.*, the reference frame of the hangar), how long does it take for the signal to reach the front of the rocket?

Answers:

(a) In the rest frame of the hangar the rocket is contracted by a factor γ, so in its own rest frame it must be longer by that factor. So

$$\boxed{\ell_R = \gamma a \ , \qquad \text{where } \gamma = \frac{1}{\sqrt{1 - v^2/c^2}} \ .}$$

(b) In the rest frame of the rocket the hangar is moving at speed v, so it is contracted. So

$$\boxed{\ell_H = a/\gamma \ .}$$

(c) The two clocks are synchronized in their own rest frame, but in the rest frame of the hangar they are moving. The pair of clocks will therefore not be synchronized in the frame of the hangar, but instead the trailing clock will read later. The clock in the tail is the trailing clock, so t_1 will be greater than t_2, and hence t_2 will be $\boxed{\text{less than } t_1.}$ This result is also required to resolve the paradox which makes the problem interesting. As seen in the rest frame of the rocket, the door closes in back of the rocket at t_1, and opens in front of the rocket at t_2. Since in this frame the rocket is too long to fit inside the hangar, it must be that the door in front opens before the door in back closes.

(d) The trailing clock always reads later by $v\ell_0/c^2$, where ℓ_0 is the separation in the rest frame. In this case $\ell_0 = \ell_R = \gamma a$, so

$$\boxed{\Delta t = t_2 - t_1 = -v\gamma a/c^2 \ .}$$

(e) In the frame of the rocket, the door on the right is already opened at the instant when the door on the left is closed:

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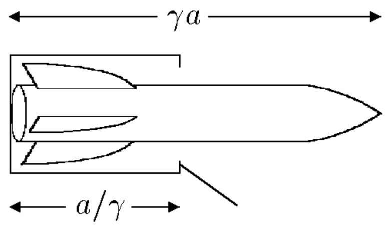
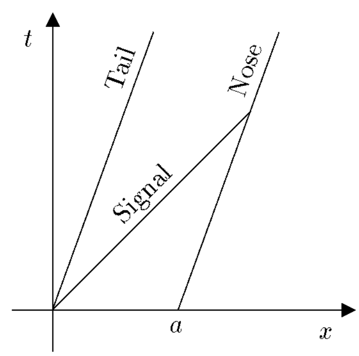


(f) Working in the frame of the rocket, we need only calculate the time it takes light to travel the length of the rocket, which from (a) is just $\ell_R = \gamma a$. Thus,

$$\boxed{\text{Time interval} = \gamma a/c\ .}$$

(g) Working in the rest frame of the hangar, we can choose the origin of time t so that the light signal is emitted at $t = 0$. In this frame the rocket is moving at speed v to the right. If we choose the origin of the x-coordinate axis so that at $t = 0$ the tail is located at $x = 0$, then the trajectory of the tail is given by

$$x_{\text{tail}} = vt\ .$$



Since the rocket has length a in this frame, the trajectory of the nose is given by

$$x_{\text{nose}} = a + vt\ .$$

The light signal is then emitted from the tail at $t = 0$, $x = 0$, so its trajectory is

$$x_{\text{sig}} = ct\ .$$

Solving for the intersection of x_{sig} and x_{nose}, one has

$$ct = a + vt \quad \Longrightarrow \quad \boxed{t = \frac{a}{c - v}\ .}$$