MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

PROBLEM SET 2 SOLUTIONS

February 10, 2005

Corrected Version, 4:45 p.m.: Typos fixed in Problem 9 (Y&F 2.14)

Velocity and Acceleration in One Dimension

Problem 1: Velocity and acceleration in one dimension

SG:1A.5 An object moves along the x-axis with constant acceleration a. Its position and velocity at time $t = 0$ are $x = x_0$ and $v = v_0$ respectively; at some later time t it has position x and velocity v. Use the definitions of velocity and acceleration to prove that

$$x = x_0 + v_0 t + \frac{1}{2} a t^2$$

and

$$v^2 = v_0^2 + 2a(x - x_0) \ .$$

[It is very easy to prove this with calculus, but for constant acceleration you don't actually *need* calculus to derive these equations.]

Answer:

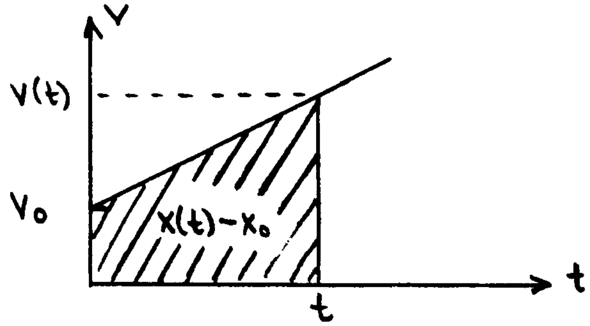
Solution with calculus:

$$a = \frac{dv}{dt} \Rightarrow v(t) - v_0 = \int_0^t dt' \, a = at \Rightarrow v(t) = v_0 + at ,$$

$$v = \frac{dx}{dt} \Rightarrow x(t) - x_0 = \int_0^t dt' \, v(t') = \int_0^t dt' \, (v_0 + at') = v_0 t + \frac{1}{2} a t^2 ,$$

$$2a(x - x_0) = 2a v_0 t + a^2 t^2 = (v_0 + at)^2 - v_0^2 = v^2 - v_0^2 \Rightarrow v^2 = v_0^2 + 2a(x - x_0)$$

Solution without calculus:



Velocity versus time

$$x(t) - x_0 = \text{Area of shaded region}$$
$$= v_0 t + \frac{1}{2}(v(t) - v_0) t$$
$$= v_0 t + \frac{1}{2} a t^2$$

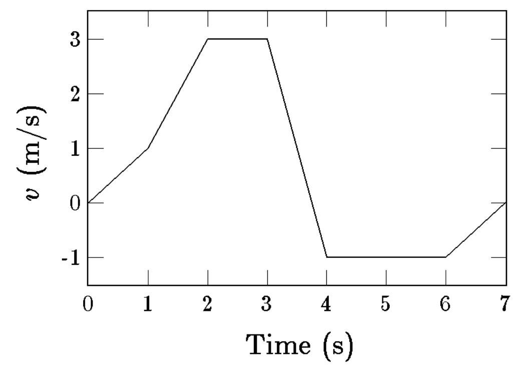
Problem 2: Interpreting a graph of v vs. t

SG:1A.6 The graph shows the velocity of a particle (along the x-axis) as a function of time.

(a) When is the acceleration of the particle (i) positive, (ii) negative, (iii) zero?

(b) What is the particle's displacement after 3.5 s? After 7 s?

(c) Describe in words the motion of the particle.



Answer:

(a) Acceleration is rate of change of velocity : $a = \frac{dv}{dt}$,
i.e. it is the slope of the curve v versus t. Hence
$a > 0$ for $0 < t < 2s$ and for $6s < t < 7s$,
$a = 0$ for $2s < t < 3s$ and for $4s < t < 6s$,
$a < 0$ for $3s < t < 4s$

(b) Displacement as a function of time: $x(t) - x_0 = \int_0^t v(t')\,dt'$
is the area under the curve of v versus t. One can add the areas on a chart:

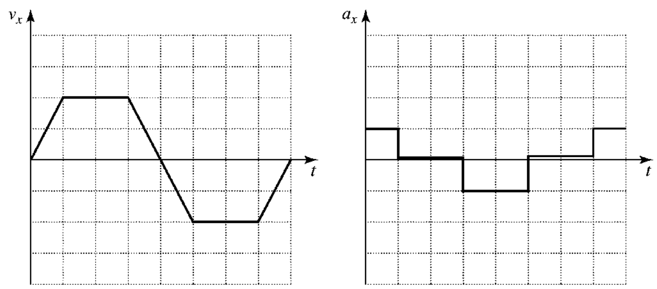
Interval	v_i (m/s)	v_f (m/s)	$V_{average}$ (m/s)	Area (m)
0-1 s	0	1	0.5	0.5
1-2 s	1	3	2	2.0
2-3 s	3	3	3	3.0
3-3.5 s	3	1	2	1.0
3.5-4 s	1	-1	0	0.0
4-6 s	-1	-1	-1	-2.0
6-7 s	-1	0	-0.5	-0.5

At $t = 3.5s$ the area is 6.5 m, and at $t = 7s$ it is 4.0 m.

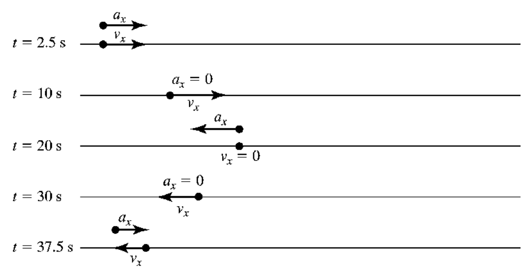
(c) *The particle starts from rest, and then moves forward. It slows down, coming to rest at 3.75 seconds after it started to move, and then it reverses its motion. At the end of 7 seconds it comes to rest again, at 4 m from its initial position.*

Problem 4 (Y&F 2.19): Graphing the motion of a spider

(a)



(b)



Problem 5 (Y&F:2.27): Acceleration and braking of the Ford Aspire

(a) speeding up:

$$x - x_0 = 1320 \text{ ft}, v_{0x} = 0, t = 19.9 \text{ s}, a_x = ?$$

$$x - x_0 = v_{0x}t + \frac{1}{2}a_x t^2 \text{ gives } a_x = 6.67 \text{ ft/s}^2 \ .$$

slowing down:

$$x - x_0 = 146 \text{ ft}, v_{0x} = 88.0 \text{ ft/s}, v_x = 0, a_x = ?$$

$$v_x = v_{0x} + 2a_x(x - x_0) \text{ gives } a_x = -26.5 \text{ ft/s}^2 \ .$$

(b)

$$x - x_0 = 1320 \text{ ft}, v_{0x} = 0, a_x = 6.67 \text{ ft/s}^2, v_x = ?$$

$$v_x = v_{0x} + 2a_x(x - x_0) \text{ gives } v_x = 133 \text{ ft/s} = 90.5 \text{ mph.}$$

$$a_x \text{ must not be constant.}$$

(c)

$$v_{0x} = 88.0 \text{ ft/s}, a_x = -26.5\text{ft/s}^2, v_x = 0, t = ?$$

$$v_x = v_{0x} + a_x t \text{ gives } t = 3.32 \text{ s.}$$

Using Vectors

Problem 6: Using vectors to trace two couples and a dog

SG:1B.2Albert, Betty, Carol, and Dave are playing frisbee in a square field whose sides happen to run due east and due north. Albert's position vector relative to one corner of the field is [10, 7, 0] m, where x is east and y north (z is up, but you can assume that the field is level).

(a) Betty is 14 m northeast of Albert, Carol is 10 m east of Betty, and Dave is 8 m south of Carol. What are the position vectors of Betty, Carol, and Dave? (Take the same corner of the field as origin for all position vectors.)

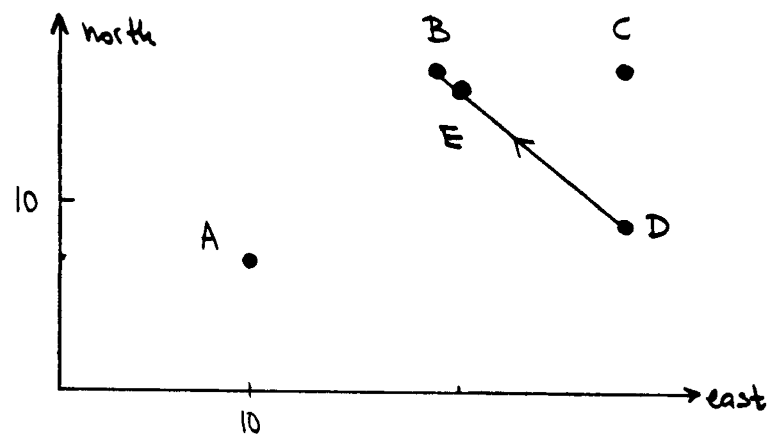
(b) How far is Albert from Carol?

(c) Dave's dog Ernie runs from Dave to Betty at 3 m/s. What is his velocity vector? How long does it take him to reach Betty? What is his position vector 4 s after leaving Dave?

(d) Make a scale drawing of the field, showing the positions of Albert, Betty, Carol, Dave, and Ernie 4 s after Ernie leaves Dave.

Answer:

(a)



$$\vec{A} = (10, 7, 0)\,m$$

$$\vec{B} = (10 + 7\sqrt{2},\ 7 + 7\sqrt{2},\ 0)\,m$$

$$\vec{C} = (20 + 7\sqrt{2},\ 7 + 7\sqrt{2},\ 0)\,m$$

$$\vec{D} = (20 + 7\sqrt{2},\ -1 + 7\sqrt{2},\ 0)\,m$$

(b) Distance AC is

$$|\vec{C}-\vec{A}| = |(10+7\sqrt{2}, 7\sqrt{2}, 0)|\,m = \sqrt{(10+7\sqrt{2})^2 + (7\sqrt{2})^2}\;m = 22.2\,m$$

(c) Displacement vector from D to B:

$\vec{d} = \vec{B} - \vec{D} = (-10, 8, 0)\,m$;

Distance from D to B: $d = |\vec{d}| = \sqrt{164}\;m = 2\sqrt{41}\;m = 12.8\,m$

Unit vector from D to B: $\hat{d} = \dfrac{\vec{d}}{d} = \dfrac{1}{\sqrt{41}}(-5, 4, 0)$

Ernie's velocity vector: $\vec{v} = v\,\hat{d}$, $v = 3\,\frac{m}{s} \Rightarrow \vec{v} = \frac{1}{\sqrt{41}}(-15, 12, 0)\frac{m}{s}$

Time to reach Betty: $t = \dfrac{d}{v} = \dfrac{2}{3}\sqrt{41}\;\dfrac{m}{s} = 4.3\,s$

Ernie's position vector after some time t: $\vec{E}(t) = \vec{D} + \vec{v}\,t$

Ernie's position after $t = 4s$:

$\vec{E}(4s) = (20+7\sqrt{2}, -1+7\sqrt{2}, 0)\,m + \dfrac{1}{\sqrt{41}}(-60, 48, 0)\,m = (21, 16, 0)\,m$

(d) See figure on previous page.

Problem 7 (Y&F:1.52): Finding angles from vectors

For all of these pairs of vectors, the angle is found from combining Equations (1.18) and (1.21), to give the angle ϕ as

$$\phi = \arccos\left(\frac{\vec{A}\cdot\vec{B}}{AB}\right) = \arccos\left(\frac{A_x B_x + A_y B_y}{AB}\right).$$

In the intermediate calculations given here, the significant figures in the dot products and in the magnitudes of the vectors are suppressed.

a) $\vec{A}\cdot\vec{B} = -22,\ A = \sqrt{40},\ B = \sqrt{13},$ **and so**

$$\phi = \arccos\left(\frac{-22}{\sqrt{40}\,\sqrt{13}}\right) = 165^\circ.$$

b) $\vec{A}\cdot\vec{B} = 60,\ A = \sqrt{34},\ B = \sqrt{136},\quad \phi = \arccos\left(\frac{60}{\sqrt{34}\sqrt{136}}\right) = 28^\circ.$

c) $\vec{A}\cdot\vec{B} = 0,\ \phi = 90.$

Velocity and Acceleration as Vectors

Problem 8: The trajectory of a kicked soccer ball

SG:1C.3A child is kicking a soccer ball in her backyard. If the ball leaves her foot with speed v_0 directed at an angle θ to the horizontal, derive expressions for the distance x that the ball travels and the height h that it reaches (assuming that it starts from $h = 0$, that the yard is level, and that air resistance is negligible).

(a) She kicks the ball with a speed of 8 m/s at an angle of 70° to the horizontal. How far from her does it hit the ground, and what maximum height does it reach? Take $g = 9.8$ m/s^2.

(b) She kicks the ball straight up and it reaches a height of 5 m. How far would it have gone horizontally if she had kicked it with the same speed, but at an angle of 45°? At what angle would she need to kick (again assuming the same speed) if she wants it to land a distance $x = 8$ m away? Draw the ball's trajectory for both possible answers. Can you do this problem if you do *not* know the value of g?

Answer:

(a)

Velocity vector of the ball : $\vec{v} = (v_x, v_y)$:

$v_x = v_0 \cos\theta$, $v_y = v_0 \sin\theta - gt$

Position of the ball : $\vec{r} = (x, y)$:

$x = \int_0^t dt' v_x = v_0 \cos\theta \, t$, $y = \int_0^t dt' v_y = v_0 \sin\theta \, t - \frac{1}{2} g t^2$

Time when the ball hits the ground, i.e. $y = 0$:

$v_0 \sin\theta \, t - \frac{1}{2} g t^2 = 0 \Rightarrow t = \frac{2}{g} v_0 \sin\theta$

Distance x at that time : $x = v_0 \cos\theta \frac{2}{g} v_0 \sin\theta = \frac{v_0^2}{g} \sin 2\theta = 4.2\,m$

Maximum height when $\frac{dy}{dt} = v_y = 0 \Rightarrow t = \frac{v_0}{g} \sin\theta$

Height at that time : $y = \frac{v_0^2}{g} \sin^2\theta - \frac{1}{2} \frac{v_0^2}{g} \sin^2\theta = \frac{1}{2} \frac{v_0^2}{g} \sin^2\theta = 2.9\,m$

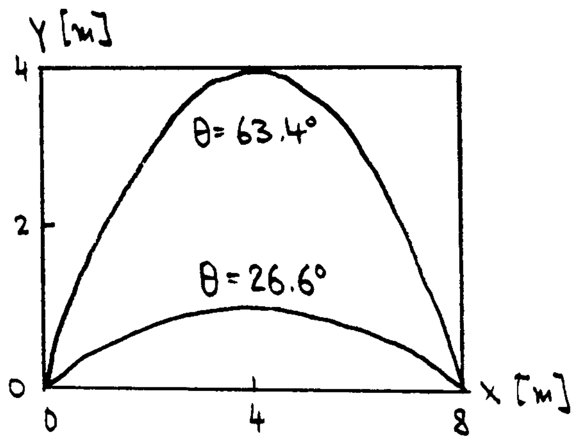
(b)

Height reached for $\theta = \frac{\pi}{2}$ (straight up) : $y = \frac{1}{2} \frac{v_0^2}{g} = 5m$.

Horizontal distance for $\theta = \frac{\pi}{4}$ (45°) : $x = \frac{v_0^2}{g} = 2y = 10\,m$.

Angle for landing at $x = 8m$: $x = \frac{v_0^2}{g} \sin 2\theta = 8m \Rightarrow \sin 2\theta = \frac{8m}{10m} = \frac{4}{5}$,

this has two solutions : $\theta = 26.6°$ and $\theta = 63.4°$.



Note that in part (b) of this problem we have not used the numerical value of g. Hence, the answers to the above questions do not depend on g.

Problem 9 (Y&F:2.14): Tracking a suspected UFO

(a) The displacement vector is:

$$\vec{\mathbf{r}}(t) = -(5.0\ \text{m/s})t\hat{\boldsymbol{\imath}} + (10.0\ \text{m/s})t\hat{\boldsymbol{\jmath}} + \left[(7.0\ \text{m/s})t - (3.0\ \text{m/s}^2)t^2\right]\hat{\boldsymbol{k}}\ .$$

The velocity vector is the time derivative of the displacement vector:

$$\frac{\mathrm{d}\vec{\mathbf{r}}(t)}{\mathrm{d}t} = (-5.0\ \text{m/s})\hat{\boldsymbol{\imath}} + (10.0\ \text{m/s})\hat{\boldsymbol{\jmath}} + \left[7.0\ \text{m/s} - 2(3.0\ \text{m/s}^2)t\right]\hat{\boldsymbol{k}}\ .$$

and the acceleration vector is the time derivative of the velocity vector:

$$\frac{\mathrm{d}^2\vec{\mathbf{r}}(t)}{\mathrm{d}t^2} = -6.0\ \text{m/s}^2\hat{\boldsymbol{k}}\ .$$

At $t = 5.0$ s:

$$\begin{aligned}
\vec{\mathbf{r}}(t) &= -(5.0\ \text{m/s})(5.0\ \text{s})\hat{\boldsymbol{\imath}} + (10.0\ \text{m/s})(5.0\ \text{s})\hat{\boldsymbol{\jmath}} + \left[7.0\ \text{m/s})(5.0\ \text{s}) - (3.0\ \text{m/s}^2)(25.0\ \text{s}^2)\right]\hat{\boldsymbol{k}} \\
&= (-25.0\ \text{m})\hat{\boldsymbol{\imath}} + (50.0\ \text{m})\hat{\boldsymbol{\jmath}} - (40.0\ \text{m})\hat{\boldsymbol{k}}\ . \\
\frac{\mathrm{d}\vec{\mathbf{r}}(t)}{\mathrm{d}t} &= (-5.0\ \text{m/s})\hat{\boldsymbol{\imath}} + (10.0\ \text{m/s})\hat{\boldsymbol{\jmath}} + \left[(7.0\ \text{m/s} - (6.0\ \text{m/s}^2)(5.0\ \text{s})\right]\hat{\boldsymbol{k}} \\
&= (-5.0\ \text{m/s})\hat{\boldsymbol{\imath}} + (10.0\ \text{m/s})\hat{\boldsymbol{\jmath}} - (23.0\ \text{m/s})\hat{\boldsymbol{k}}\ . \\
\frac{\mathrm{d}^2\vec{\mathbf{r}}(t)}{\mathrm{d}t^2} &= -6.0\ \text{m/s}^2\hat{\boldsymbol{k}}\ .
\end{aligned}$$

(b) The velocity in both the x- and the y-directions is constant and nonzero; thus the overall velocity can never be zero.

(c) The object's acceleration is constant, since t does not appear in the acceleration vector.

Problem 10 (Y&F:3.12): A daring jump

Time to fall 9.00 m from rest:

$$\begin{aligned}
y &= \frac{1}{2}gt^2 \\
9.00\,\text{m} &= \frac{1}{2}(9.8\,\text{m/s}^2)t^2 \\
t &= 1.36\,\text{s}
\end{aligned}$$

Speed to travel 1.75 m horizontally:

$$\begin{aligned}
x &= v_0 t \\
1.75\,\text{m} &= v_0(1.36\,\text{s}) \\
v_0 &= 1.3\,\text{m/s}
\end{aligned}$$

Problem 11 (Y&F:3.24): The trajectory of water from a fire hose

a)

$$v_0 \cos\alpha t = 45.0\,\text{m}$$

$$\cos\alpha = \frac{45.0\,\text{m}}{(25.0\,\text{m/s})(3.00\,\text{s})} = 0.600$$

$$\alpha = 53.1°$$

b)

$$v_x = (25.0\,\text{m/s})\cos 53.1° = 15.0\text{ m/s}$$
$$v_y = 0$$
$$v = 15.0\,\text{m/s}$$
$$a = 9.80\,\text{m/s}^2\text{ downward}$$

c) Find y when $t = 3.00\,\text{s}$

$$y = v_0 \sin\alpha t - \frac{1}{2}gt^2$$
$$= (25.0\,\text{m/s})(\sin 53.1°)(3.00\text{s}) - \frac{1}{2}(9.80\,\text{m/s}^2)(3.00\,\text{s})^2$$
$$= 15.9\,\text{m}$$
$$v_x = 15.0\text{ m/s} = \text{constant}$$
$$v_y = v_0 \sin\alpha - gt = (25.0\,\text{m/s})(\sin 53.1°) - (9.80\,\text{m/s}^2)(3.00\,\text{s}) = -9.41$$
$$v = \sqrt{v_x{}^2 + v_y{}^2} = \sqrt{(15.0\,\text{m/s})^2 + (-9.41\,\text{m/s}^2} = 17.7\text{ m/s}$$

Circular Motion

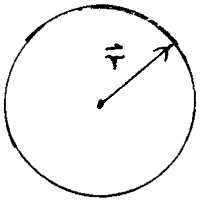
Problem 12: Centripetal acceleration of geosynchronous satellites

SG:1D.3 (a) A geosynchronous or geostationary satellite is so called because it takes 24 hours to complete one orbit. Such satellites orbit at a height of 35,800 km above the Earth's surface. What is the centripetal acceleration of geosynchronous satellites? [The radius of the Earth is 6,400 km.]

(b) What is the centripetal acceleration of a point on the Earth's equator, at sea level?

Compare your answers to both parts of the question with the value of g at sea level (9.8 m/s^2).

(a)



Circular motion: $\vec{r}(t) = r(\cos(\omega t), \sin(\omega t))$,

for geosynchronous satellite: $\omega = 2\pi/\text{day}$

Velocity vector: $\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = r\omega(-\sin(\omega t), \cos(\omega t))$

Acceleration vector: $\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = -r\omega^2(\cos(\omega t), \sin(\omega t))$

Centripetal acceleration: $a = |\vec{a}(t)| = r\omega^2$

Radius of earth: R, height above surface: h $\Rightarrow$ $r = R + h = 42{,}200$ km $\Rightarrow$

$$a = 42.2 \cdot 10^6\,\text{m}\,\frac{4\pi^2}{(24 \cdot 60 \cdot 60\,\text{s})^2} = 0.22\,\frac{\text{m}}{\text{s}^2} \ll g = 9.8\,\frac{\text{m}}{\text{s}^2}$$

(b) Now $h = 0$ such that

$$a = R\omega^2 = 6.4 \cdot 10^6\,\text{m}\,\frac{4\pi^2}{(24 \cdot 60 \cdot 60\,\text{s})^2} = 0.034\,\frac{\text{m}}{\text{s}^2} \ll g = 9.8\,\frac{\text{m}}{\text{s}^2}$$

Problem 13 (Y&F:3.29): Acceleration of objects on the Earth's surface

Using the given values in Eq. (3.30),

$$a_{\text{rad}} = \frac{4\pi^2(6.38\times10^6\ \text{m})}{((24\,\text{h})(3600\,\text{s/h}))^2} = 0.034\ \text{m/s}^2 = 3.4\times10^{-3}\,g.$$

(Using the time for the siderial day instead of the solar day will give an answer that differs in the third place.) b) Solving Eq. (3.30) for the period T with $a_{\text{rad}} = g$,

$$T = \sqrt{\frac{4\pi^2(6.38\times10^6\,\text{m})}{9.80\,\text{m/s}^2}} = 5070\,\text{s} \sim 1.4\,\text{h}.$$

Frames of Reference and Relative Velocity

Problem 15: (Y&F:3.39): A canoe on a moving river

The velocity components are

$$-0.50\,\text{m/s} + (0.40\,\text{m/s})/\sqrt{2}\,\text{east} \text{ and } (0.40\,\text{m/s})/\sqrt{2}\,\text{south},$$

for a velocity relative to the earth of 0.36 m/s, 52.5° south of west.