MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

PROBLEM SET 6 SOLUTIONS

March 10, 2005

Energy Conservation Basics:

Problem 1 (SG): *STUDY:* Energy conservation riddles

4D.1 (S)

(a) John is holding a 17 inch computer monitor, with a mass of 25 kg, at a constant height. He complains that it is hard work, and he is becoming exhausted. Jim, who is taking physics, tells him that since there is no displacement in the direction of the applied force, he is not doing any work, and therefore should not be tired. Who is right, and why?

(b) Mary is riding an elevator from the 4th to the 7th floor, wearing a backpack of mass M. Between the 5th and 6th floors the elevator is moving at constant speed, through a distance ℓ. Joan, another physics student, argues that since the velocity of Mary's backpack is constant, the total force must be zero, and therefore Mary is applying an upward force just enough to cancel that of gravity, Mg. Since the displacement is upward by a distance ℓ, the work that Mary has done on the backpack is $W = Mg\ell$. Mary, on the other hand, points out that the weight she feels is just the same as it would be if she were stationary on the sidewalk, in which case there would be no displacement and therefore no work. Since she is burning no more calories than she would if she were on the sidewalk, and since energy cannot be created from nothing, there is no energy available for her to do work on the backpack. Who is right, and why?

(c) A skater on a frictionless ice rink is initially stationary. Holding onto a rope attached to the wall, he gives a yank and starts himself moving toward the wall. Jean, a physics student who is watching, tells her friend Joe that since the kinetic energy of the skater has increased, work must have been done. It was the rope that applied the force, Jean explains, so it was the rope that did the work on the skater. "Nonsense,“ Joe replied, "ropes can't do work! Ropes don't have any source of energy, so the principle of energy conservation implies that they can't do work. Obviously it was the skater who did the work." "But the skater can't possibly exert a net force on himself,“ retorted Jean, "so he can't have done the work." Who is right here, and why?

Answer: See complete solution in the *Study Guide*.

Problem 2 (SG): Potential energy of a spring

4B.3 Suppose that a spring obeys Hooke's law with spring constant k. Show that the potential energy stored in it when it is compressed (or extended) by an amount x is given by $\frac{1}{2}kx^2$, taking the potential energy of the unstretched spring to be zero.

Answer:

The general formula for potential energy for a force in one dimension is

$$U(x) = U(0) - \int_0^x F_x(x')\,\mathrm{d}x' \ ,$$

where Hooke's law for an ideal spring implies that $F_x = -kx$. Therefore

$$U(x) = U(0) + \int_0^x kx'\,\mathrm{d}x' = U(0) + \frac{1}{2}kx^2 \ .$$

If we take the potential energy $U(0)$ of the unstretched spring to be zero, we see that the potential energy stored in the spring when it is stretched by a distance x is $\boxed{\tfrac{1}{2}kx^2.}$

Problem 3 (SG): *STUDY:* Energy considerations in lifting a box

4B.2 (S) You lift a box of mass m from the floor and place it on a shelf a height h above floor level.

(a) How much work have you done on the box? What is the total work done on the box? What happens to the energy of the box? Do the details of the motion of the box during the lift make a difference to your answers?

(b) At the instant when the box has reached height $\frac{1}{2}h$, is the work you have done on it so far more or less than half of the total work you will do during the lift? Explain your reasoning.

Answer: See complete solution in the *Study Guide.*

Problem 4 (Y&F): Spring, block, and table top with friction

7.43: The initial and final kinetic energies are both zero, so the work done by the spring is the negative of the work done by friction, or $\frac{1}{2}kx^2 = \mu_k mgl,$ where l is the distance the block moves. Solving for μ_k,

$$\mu_k = \frac{(1/2)kx^2}{mgl} = \frac{(1/2)(100\ \text{N/m})(0.20\ \text{m})^2}{(0.50\ \text{kg})(9.80\ \text{m/s}^2)(1.00\ \text{m})} = 0.41.$$

Problem 5 (SG): *STUDY:* Firing a spring gun into the air

4D.5 (S) A spring with spring constant k is used to power a spring gun which launches a ball of mass m vertically upwards. If the spring is compressed by an amount x beyond its equilibrium position to launch the ball, derive an expression for the maximum height reached by the ball, measured from its position when the spring is compressed. If k is 200 N/m, m = 50 g, and the spring is initially compressed by 4 cm, how high will the ball go?

Answer: See complete solution in the *Study Guide.*

Problem 6 (Y&F): Inclined plane with a spring

7.74: a) From either energy or force considerations, the speed before the block hits the spring is

$$\begin{aligned} v &= \sqrt{2gL(\sin\theta - \mu_k\cos\theta)} \\ &= \sqrt{2(9.80\ \mathrm{m/s^2})(4.00\ \mathrm{m})(\sin 53.1° - (0.20)\cos 53.1°)} \\ &= 7.30\ \mathrm{m/s}. \end{aligned}$$

b) This does require energy considerations; the combined work done by gravity and friction is $mg(L+d)(\sin\theta - \mu_k\cos\theta)$, and the potential energy of the spring is $\frac{1}{2}kd^2$, where d is the maximum compression of the spring. This is a quadratic in d, which can be written as

$$d^2\frac{k}{2mg(\sin\theta - \mu_k\cos\theta)} - d - L = 0.$$

The factor multiplying d^2 is $4.504\ \mathrm{m^{-1}}$, and use of the quadratic formula gives $d = 1.06\ \mathrm{m}$. c) The easy thing to do here is to recognize that the presence of the spring determines d, but at the end of the motion the spring has no potential energy, and the distance below the starting point is determined solely by how much energy has been lost to friction. If the block ends up a distance y below the starting point, then the block has moved a distance $L+d$ down the incline and $L+d-y$ up the incline. The magnitude of the friction force is the same in both directions, $\mu_k mg\cos\theta$, and so the work done by friction is $-\mu_k(2L+2d-y)mg\cos\theta$. This must be equal to the change in gravitational potential energy, which is $-mgy\sin\theta$. Equating these and solving for y gives

$$y = (L+d)\frac{2\mu_k\cos\theta}{\sin\theta + \mu_k\cos\theta} = (L+d)\frac{2\mu_k}{\tan\theta + \mu_k}.$$

Using the value of d found in part (b) and the given values for μ_k and θ gives $y = 1.32\ \mathrm{m}$.

Problem 7 (SG): Potential energy of gravitation and electrostatics

4B.5 (H) Derive expressions for, and plot, the potential energy

(a) of a space probe of mass m as a function of distance from the Sun, which has mass M (neglect the effect of the planets);

(b) of a positive charge q as a function of distance from a positive charge Q;

(c) of a negative charge $-q$ as a function of distance from a positive charge Q.

It is convenient to be able to draw the plots in a way that is independent of the specific values of M, m, Q, and q. To do this, introduce an arbitrary reference distance r_0. In case (a) the scale on your y-axis should be in terms of GMm/r_0, while in cases (b) and (c) the scale is in terms of $Qq/(4\pi\epsilon_0 r_0)$. In all cases the horizontal axis can be taken as r/r_0, and U_0 should be defined so that U is zero when the distance is infinitely large. For simplicity you may treat this as a one-dimensional problem, considering motion only along a line joining the two objects. (The correct answer to the one-dimensional problem is also valid in three dimensions, but you are not asked to show this.)

Answer:

Derivations:

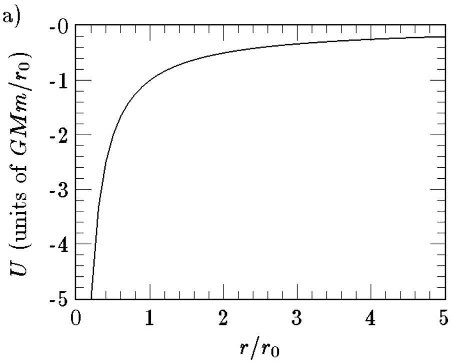
(a) $$U(r_p) = U_0 - \int_\infty^{r_p} F\,\mathrm{d}r = -\int_\infty^{r_p}\left[-\frac{GMm}{r^2}\right]\mathrm{d}r = -\frac{GMm}{r}\bigg|_\infty^{r_p} = -\frac{GMm}{r_p}\ .$$

(b) $$U(r_p) = U_0 - \int_\infty^{r_p} F\,\mathrm{d}r = -\int_\infty^{r_p}\left[\frac{1}{4\pi\epsilon_0}\frac{Qq}{r^2}\right]\mathrm{d}r = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r}\bigg|_\infty^{r_p} = \frac{1}{4\pi\epsilon_0}\frac{Qq}{r_p}\ .$$

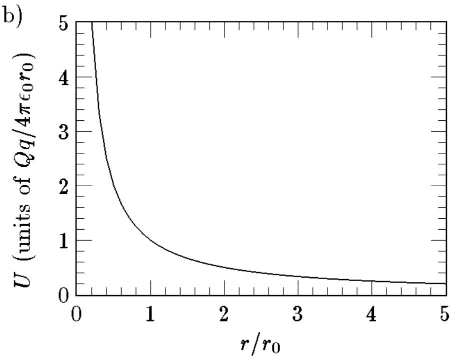
(c) Same as (b), but with opposite sign, so $U(r_p) = -\dfrac{1}{4\pi\epsilon_0}\dfrac{Qq}{r_p}$.

Graphs:

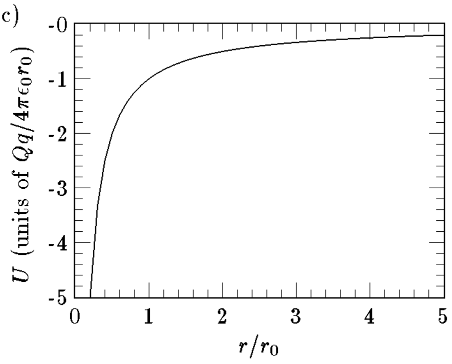
(a)



(b)



(c)



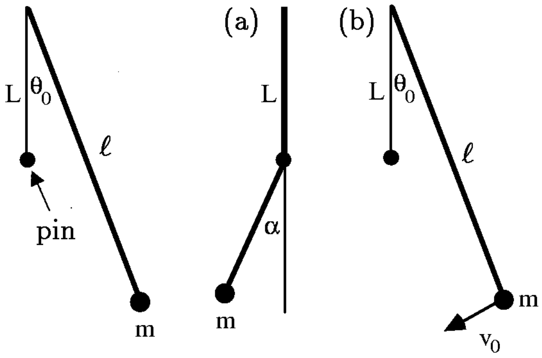
Energy Conservation and Circular Motion:

Problem 8 (SG): The pin and the pendulum

4D.4 (H) A simple pendulum consisting of a mass m attached to a string of length ℓ is released from rest at an angle θ_0. A pin is located at a distance L below the pivot point. When the pendulum swings down, the string hits the pin as shown.

(a) What is the maximum angle α that the string makes with the vertical after hitting the pin?

(b) If the bob had been released with an initial velocity v_0 as shown, what would be the maximum value of α? How would this be affected if v_0 were in the opposite direction?



Answer:

(a) In this problem mechanical energy (i.e., kinetic energy plus potential energy) is conserved. No work is done when the string hits the pin and partially wraps around it, because the part of the string that experiences a force is the part that is in contact with the pin, and is stationary. Work can be done on an object only if the object moves.

Let us adopt a coordinate system with the y-axis vertical, and with $y = 0$ at the top end of the string. We can also choose the gravitational potential energy at $y = 0$ to be zero, so the potential energy of the pendulum bob is given by $U = mgy$. Since the string is massless, the only energy of the pendulum is that of the bob. Since the bob is initially at rest, its initial mechanical energy is solely its potential energy:

$$E_{\text{initial}} = U(\theta_0) = -mg\ell\cos\theta_0 \ .$$

When the bob reaches its maximum height on the left side it will again be stationary, so again the only mechanical energy will be potential:

$$E_{\text{final}} = U_{\text{pin}}(\alpha_{\text{max}}) = -mg[L + (\ell - L)\cos\alpha_{\text{max}}] \ .$$

(Note that although $U_{\text{pin}}(\alpha_{\text{max}})$ and $U(\theta_0)$ both represent the same physical quantity, the potential energy of the bob, they do so under different circumstances. $U_{\text{pin}}(\alpha_{\text{max}})$ applies when the string hits the pin, and $U(\theta_0)$ applies when it does not. They therefore have different mathematical forms, so they are different mathematical functions. I have chosen to distinguish them by using a subscript for one and no subscript for the other. It would be bad notation to omit the subscript and call them $U(\alpha_{\text{max}})$ and $U(\theta_0)$, because once $U(\theta_0)$ has been defined by $U(\theta_0) = -mg\ell\cos\theta_0$, then the standard conventions of mathematics imply that $U(\alpha_{\text{max}})$ should mean $-mg\ell\cos\alpha_{\text{max}}$.) By conservation of energy,

$$E_{\text{final}} = E_{\text{initial}} \quad \Longrightarrow \quad \ell\cos\theta_0 = L + (\ell - L)\cos\alpha_{\text{max}} \ ,$$

so

$$\boxed{\alpha_{\text{max}} = \cos^{-1}\left\{\frac{\ell\cos\theta_0 - L}{\ell - L}\right\} .}$$

(b) If the bob has an initial speed v_0, then its initial mechanical energy includes a kinetic contribution:

$$E_{\text{initial}} = \frac{1}{2}mv_o^2 - mg\ell\cos\theta_0 .$$

The expression for E_{final} does not change form, so

$$E_{\text{final}} = E_{\text{initial}} \quad\Longrightarrow\quad \frac{1}{2}mv_o^2 - mg\ell\cos\theta_0 = -mg[L + (\ell - L)\cos\alpha_{\text{max}}] ,$$

or

$$\ell\cos\theta_0 - \frac{1}{2}\frac{v_0^2}{g} = L + (\ell - L)\cos\alpha_{\text{max}} ,$$

so

$$\boxed{\alpha_{\text{max}} = \cos^{-1}\left\{\frac{\ell\cos\theta_0 - L - \frac{v_o^2}{2g}}{\ell - L}\right\} .}$$

The initial mechanical energy does not depend on the direction of v_0, so $\boxed{\alpha_{\text{max}} \text{ will not depend on the direction of the initial velocity.}}$

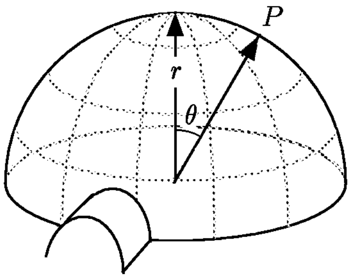
Problem 9 (SG): Sliding down an igloo

7.3 (H) An Eskimo child is using her parents' hemispherical igloo as a slide. She starts off from rest at the top and slides down under the influence of gravity. The surface of the igloo is effectively frictionless.

(a) What is her potential energy at point P (see diagram)? Define the potential energy so that it is zero when the child is on the ground.

(b) Draw a force diagram for her at point P.

(c) Does she remain in contact with the igloo all the way to the ground? If not, at what angle θ does she lose contact?



Answer:

(a) In general

$$U = Mgh ,$$

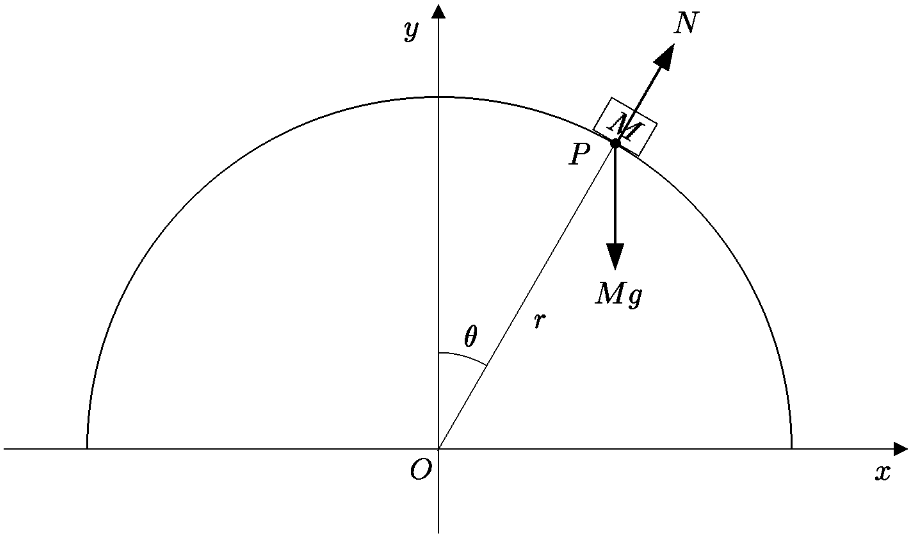
where h is the height above the zero of potential, which in this case is the ground. Since

$$h = r\cos\theta ,$$

we have

$$\boxed{U = Mgr\cos\theta\ .}$$

(b)



(c) She will remain in contact as long as the normal force is positive, but she will lose contact if it ever falls to zero. To find the normal force, balance forces in the radial direction. To know the radial acceleration, we must know v, which can be found by energy conservation:

$$Mgr\cos\theta + \frac{1}{2}Mv^2 = Mgr \quad\Longrightarrow\quad v^2 = 2gr(1-\cos\theta)\ .$$

The radial component of the $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ equation reads:

$$N - Mg\cos\theta = -M\frac{v^2}{r}\ ,$$

so

$$\begin{aligned} N &= Mg\cos\theta - 2Mg(1-\cos\theta) \\ &= Mg(3\cos\theta - 2)\ . \end{aligned}$$

So $N = 0$ when $3\cos\theta - 2 = 0$, at which point the child would lose contact. Finally,

$$\boxed{\text{Child loses contact at } \theta = \cos^{-1}\left(\frac{2}{3}\right)\ .}$$

Energy Diagrams:

Problem 10 (Y&F): Equililibrium points in an energy diagram

7.38: a) Considering only forces in the x-direction, $F_x = -\frac{dU}{dx}$, and so the force is zero when the slope of the U *vs* x graph is zero, at points b and d. b) Point b is at a potential minimum; to move it away from b would require an input of energy, so this point is stable. c) Moving away from point d involves a decrease of potential energy, hence an increase in kinetic energy, and the marble tends to move further away, and so d is an unstable point.

More Advanced Energy Conservation Problems:

Problem 11 (Y&F): Nonconservative force acting on an electron

7.28: a) From (0, 0) to (0, L), $x = 0$ and so $\vec{F} = \mathbf{0}$,, and the work is zero. From (0, L) to (L, L), $\vec{F}$ and $d\vec{l}$ are perpendicular, so $\vec{F} \cdot d\vec{l} = 0$. and the net work along this path is zero. b) From (0, 0) to (L, 0), $\vec{F} \cdot d\vec{l} = 0$. From ($L$, 0) to ($L$, L), the work is that found in the example, $W_2 = CL^2$, so the total work along the path is CL^2. c) Along the diagonal path, $x = y$, and so $\vec{F} \cdot d\vec{l} = Cy\,dy$; integrating from 0 to L gives $\frac{CL^2}{2}$. (It is not a coincidence that this is the average to the answers to parts (a) and (b).) d) The work depends on path, and the field is not conservative.

Problem 12 (Y&F): Determining potential energy from observed motions

Answer:

(a) Starting with the trajectory equations,

$$x = x_0 \cos \omega_0 t \ , \qquad y = y_0 \sin \omega_0 t \ ,$$

one finds

$$F_x = ma_x = m \frac{\mathrm{d}^2 x}{\mathrm{d}t^2} = -m\omega_0^2 x_0 \cos \omega_0 t = \boxed{-m\omega_0^2 \, x \ ,}$$

and

$$F_y = ma_y = m \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = -m\omega_0^2 y_0 \sin \omega_0 t = \boxed{-m\omega_0^2 \, y \ .}$$

(b) The general expression for the potential energy function, given the force, is

$$U(\vec{\mathbf{r}}_p) \equiv U_0 - \int_{\vec{\mathbf{r}}_0}^{\vec{\mathbf{r}}_p} \vec{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}} \ .$$

Here U_0 represents the potential energy at $\vec{\mathbf{r}}_0$, which is an arbitrary reference point. For this problem we are told to take $U = 0$ at $x = 0$ and $y = 0$, so we take this point as $\vec{\mathbf{r}}_0$, with $U_0 = 0$. For a conservative force the integral in the above equation is independent of the path, so we can take whatever path seems the simplest. I will choose to integrate from $(x, y) = (0, 0)$ to $(x, 0)$ along a straight line, and then from $(x, 0)$ to (x, y) along a straight line. This gives

$$\begin{aligned} U(x,y) &= -\int_0^x F_x(x',0)\,\mathrm{d}x' - \int_0^y F_y(x,y')\,\mathrm{d}y' \\ &= m\omega_0^2 \int_0^x x'\,\mathrm{d}x' + m\omega_0^2 \int_0^y y'\,\mathrm{d}y' \\ &= \boxed{\frac{1}{2} m\omega_0^2 \left(x^2 + y^2\right) \ .} \end{aligned}$$

(c) (i) The velocity is given in general by

$$\begin{aligned} v_x &= \frac{\mathrm{d}x}{\mathrm{d}t} = -\omega_0 x_0 \sin\omega_0 t = -\frac{\omega_0 x_0}{y_0}\, y \ , \\ v_y &= \frac{\mathrm{d}y}{\mathrm{d}t} = \omega_0 y_0 \cos\omega_0 t = \frac{\omega_0 y_0}{x_0}\, x \ . \end{aligned}$$

For $x = x_0$, $y = 0$, we have

$$E_k = \frac{1}{2} m \left(v_x^2 + v_y^2\right) = \frac{1}{2} m \left[0 + (\omega_0 y_0)^2\right] = \frac{1}{2} m\omega_0^2 y_0^2 \ ,$$

and

$$U = \frac{1}{2} m\omega_0^2 \left(x^2 + y^2\right) = \frac{1}{2} m\omega_0^2 x_0^2 \ .$$

So

$$E_{\text{tot}} = E_k + U = \boxed{\frac{1}{2} m\omega_0^2 (x_0^2 + y_0^2) \ .}$$

(ii) For $x = 0$, $y = y_0$, we have

$$E_k = \frac{1}{2} m \left(v_x^2 + v_y^2\right) == \frac{1}{2} m \left[(-\omega_0 x_0)^2 + 0\right] = \frac{1}{2} m\omega_0^2 x_0^2 \ ,$$

and

$$U = \frac{1}{2} m\omega_0^2 \left(x^2 + y^2\right) = \frac{1}{2} m\omega_0^2 y_0^2 \ .$$

So again,

$$E_{\text{tot}} = \boxed{\frac{1}{2} m\omega_0^2 (x_0^2 + y_0^2) \ .}$$

Since mechanical energy is converved for a conservative force, it is not surprising that this is the same answer as for case (i).

Problem 13 (SG): *STUDY:* Deriving mgh from the general formula for spheres

4B.4 (S) The gravitational potential energy of a mass m outside a spherical body of mass M is given by $U(r) = -GMm/r$, where r is the distance of the mass m from the center of the body M. Show that this expression can be reduced to the form $U = mgh$ for a mass m near the Earth's surface.

Answer: See complete solution in the *Study Guide*.

Other Applications of Energy Conservation:

Problem 14 (Y&F): Hydroelectric power

7.80: (a) Stored energy $= mgh = (\rho V)gh = \rho A(1\text{ m})gh$

$$= (1000\text{ kg/m}^3)(3.0\times10^6\text{ m}^2)(1\text{ m})(9.8\tfrac{\text{m}}{\text{s}^2})(150\text{ m})$$

$$= 4.4\times10^{12}\text{ J.}$$

(b) 90% of the stored energy is converted to electrical energy, so

$$(0.90)(mgh) = 1000\text{ kW h}$$

$$(0.90)\rho V\,gh = 1000\text{ kW h}$$

$$V = \frac{(1000\text{ kW h})(\frac{3600\text{ s}}{1\text{ h}})}{(0.90)(1000\text{ kg/m}^3)(150\text{ m})(9.8\text{ m/s}^2)}$$

$$= 2.7\times10^3\text{ m}^3$$

Change in level of the lake**:**

$$A\Delta h = V_{\text{water}}$$

$$\Delta h = \frac{V}{A} = \frac{2.7\times10^3\text{ m}^3}{3.0\times10^6\text{ m}^2} = 9.0\times10^{-4}\text{ m}$$

Solutions written by Alan Guth.