MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

PROBLEM SET 8 SOLUTIONS

March 31, 2005

Rotational Kinematics:

Problem 1 (SG): Motion of a CD with constant angular acceleration

8A.1 At the start of play, a compact disc is spinning at about 700 revolutions per minute; by the end of the disc, one hour later, it has slowed to 200 revolutions per minute. What is its average angular acceleration?

(a) -8.3 rad/s^2; (b) -0.0023 rad/s^2; (c) -0.87 rad/s^2; (d) none of these.

If the angular acceleration is *constant*, what is the total angle through which any point on the CD has turned during the hour? .

(a) 2.7×10^4 rad; (b) 1.7×10^5 rad; (c) 9.7×10^6 rad; (d) none of these.

Answer: The initial angular velocity is

$$\omega_1 = 700\frac{\text{rev}}{\text{min}} \times \frac{2\pi \text{ rad}}{\text{rev}} \times \frac{\text{min}}{60 \text{ s}} = 73.3 \text{ rad/s} .$$

The final angular velocity is $\omega_2 = 200$ rev/min $= 20.7$ rad/s. The average angular acceleration is then

$$\alpha_{\text{av}} \equiv \frac{\omega_2 - \omega_1}{\Delta t} = \frac{(20.7 - 73.3) \text{ rad/s}}{3600 \text{ s}} = -0.0146 \text{ rad/s}^2 .$$

So the correct answer is **(d), none of these.**

For constant angular acceleration, the angle turned is given by $\theta = \theta_0 + \omega_0\, t + \frac{1}{2}\alpha t^2$. So in this case

$$\theta = 0 + 73.3 \,\frac{\text{rad}}{\text{s}}\,(3600 \text{ s}) - \frac{1}{2}(0.0146 \text{ rad/s}^2)\,(3600 \text{ s})^2 = 1.7 \times 10^5 \text{ rad} .$$

So the answer is **(b).**

Problem 2 (Y&F): Motion with non-constant angular acceleration

9.66: a) By successively integrating Equations (9.5) and (9.3),

$$\omega_z = \gamma t - \frac{\beta}{2}t^2 = (1.80\ \text{rad/s}^2)t - (0.125\ \text{rad/s}^3)t^2,$$

$$\theta = \frac{\gamma}{2}t^2 - \frac{\beta}{6}t^3 = (0.90\ \text{rad/s}^2)t^2 - (0.042\ \text{rad/s}^3)t^3.$$

b) The maximum positive angular velocity occurs when $\alpha_z = 0$, $t = \frac{\gamma}{\beta}$, the angular velocity at this time is

$$\omega_z = \gamma\left(\frac{\gamma}{\beta}\right) - \frac{\beta}{2}\left(\frac{\gamma}{\beta}\right)^2 = \frac{1}{2}\frac{\gamma^2}{\beta} = \frac{1}{2}\frac{(1.80\ \text{rad/s}^2)^2}{(0.25\ \text{rad/s}^3)} = 6.48\ \text{rad/s}.$$

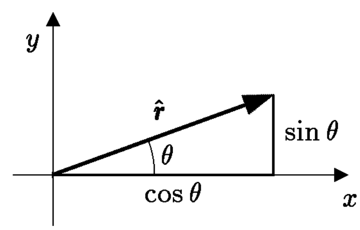
The maximum angular displacement occurs when $\omega_z = 0$, at time $t = \frac{2\gamma}{\beta}$ ($t = 0$ is an inflection point, and $\theta(0)$ is not a maximum) and the angular displacement at this time is

$$\theta = \frac{\gamma}{2}\left(\frac{2\gamma}{\beta}\right)^2 - \frac{\beta}{6}\left(\frac{2\gamma}{\beta}\right)^3 = \frac{2}{3}\frac{\gamma^3}{\beta^2} = \frac{2}{3}\frac{(1.80\ \text{rad/s}^2)^3}{(0.25\ \text{rad/s}^3)^2} = 62.2\ \text{rad}.$$

Problem 3 (SG): *STUDY:* Acceleration of a point on a rotating object

8A.2 (S) (a) Since the concepts of radial and tangential directions are used so frequently in describing rotational motion, it is sometimes useful to define unit vectors in these directions. These unit vectors are peculiar, however, because their directions depend upon the position of the particle under discussion. For a particle located in the xy-plane at an angle θ counterclockwise from the x-axis, a unit vector in the radial direction can be written as

$$\hat{\boldsymbol{r}}(\theta) = [\cos\theta, \sin\theta, 0]\ .$$



Find the corresponding expression for the unit (counterclockwise) tangential vector $\hat{\boldsymbol{u}}_\perp(\theta)$.

(b) Show that the derivatives of these unit vectors are given by

$$\frac{d\hat{\boldsymbol{r}}(\theta)}{d\theta} = \hat{\boldsymbol{u}}_\perp(\theta)$$

$$\frac{d\hat{\boldsymbol{u}}_\perp(\theta)}{d\theta} = -\hat{\boldsymbol{r}}(\theta)\ .$$

(c) Use these results to prove the formula given in the *Essentials* for the acceleration of a particle on a rigid body rotating about a fixed axis. Specifically, show that the radial and tangential components of the acceleration are given, respectively, by $a_r = -v^2/R = -R\omega^2$ and $a_\perp = R\alpha$, where ω is the angular velocity, α is the angular acceleration, and R is the distance of the particle from the axis of rotation.

Answer: See complete solution in the *Study Guide.*

Moment of Inertia:

Problem 4 (SG): *STUDY:* Proof of the parallel and perpendicular axis theorems

8C.3 (S) Prove the parallel-axis and perpendicular-axis theorems—i.e. (a) prove that the moment of inertia of a rigid body of mass M about an axis through its center of mass is related to the moment of inertia $I_{||}$ about any axis parallel to the first by the formula $I_{||} = I_{\rm cm} + Md^2$, where d is the distance between the two axes; and (b) prove that for a thin flat object in the xy-plane, the moment of inertia about the z-axis is equal to the sum of the moments of inertia about the x- and y-axes.

Answer: See complete solution in the *Study Guide*.

Problem 5 (Y&F): Comparing moments of inertia of simple objects

9.78: Quantitatively, from Table (9.2), $I_A = \frac{1}{2}mR^2, I_B = mR^2$ and $I_C = \frac{2}{3}mR^2 \cdot$ a) Object A has the smallest moment of inertia because, of the three objects, its mass is the most concentrated near its axis. b) Conversely, object B's mass is concentrated farthest from its axis. c) Because $I_{\text{sphere}} = 2/5\,mR^2$, the sphere would replace the disk as having the smallest moment of inertia.

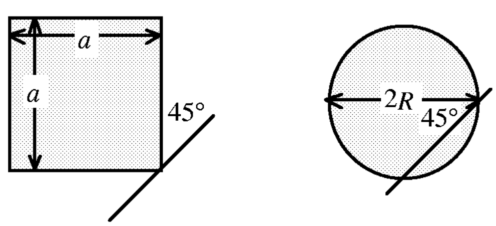
Problem 6 (SG): *STUDY:* Moments of inertia of rods, disks, and spheres (integration, and parallel and perpendicular axis theorems)

8C.4 (S) Calculate the moments of inertia of (a) a thin rod about an axis through its center and perpendicular to the rod; (b) a thin flat disk about an axis through its center and perpendicular to its plane; (c) a solid sphere about any diameter. Then deduce the moments of inertia of (d) a thin rod about an axis at one end; (e) a thin flat disk about any diameter; (f) a hollow spherical shell. Assume all the objects are of uniform density and each has mass M.

Answer: See complete solution in the *Study Guide*.

Problem 7 (SG): (H) Moments of inertia of a flat square and a flat disk

8C.5 (H) Calculate the moment of inertia of the following shapes about the specified axes (shown by a thick line). In each case assume that the object in question is a thin sheet of mass M. Note that you do *not* need to do any integration to solve this problem (but you do need the table at the end of the *Essentials).*

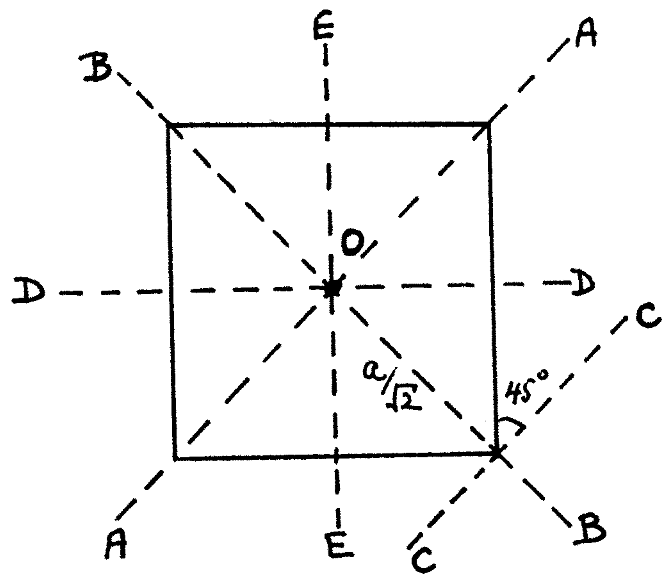


If I apply a torque $\boldsymbol{\tau}$ with respect to the specified axis to each of these objects, what will be the angular acceleration in each case?

Answer:

In this problem we will repeatedly use the parallel and perpendicular axes theorems.

(i) Consider the moment of inertia of the square about all the axes illustrated below. I_{OO} is the moment of inertia about an axis perpendicular to the square through the point O.



By the perpendicular axis theorem,

$$I_{OO} = I_{AA} + I_{BB} \ ,$$

and by symmetry $I_{AA} = I_{BB}$, so $I_{OO} = 2I_{AA}$. Similarly,

$$I_{OO} = I_{DD} + I_{EE} = 2I_{DD} \ ,$$

so

$$I_{AA} = I_{DD} \ . \qquad (1)$$

Now $I_{DD} = 2\times$ moment of inertia of half the square about I_{DD}, which can be found as the second item listed in the table on p. 277. Thus,

$$I_{DD} = 2 \times \frac{1}{3} m\ell^2 = 2 \times \frac{1}{3}\left(\frac{M}{2}\right)\left(\frac{a}{2}\right)^2 = \frac{1}{12} Ma^2 \ . \qquad (2)$$

By the parallel axis theorem,

$$I_{CC} = I_{AA} + M\left(\frac{a}{\sqrt{2}}\right)^2 = I_{AA} + \frac{1}{2} Ma^2 \ . \qquad (3)$$

From Eqs. (1), (2), and (3),

$$I_{CC} = \frac{1}{12} Ma^2 + \frac{1}{2} Ma^2$$
$$= \boxed{\frac{7}{12} Ma^2 \ .}$$

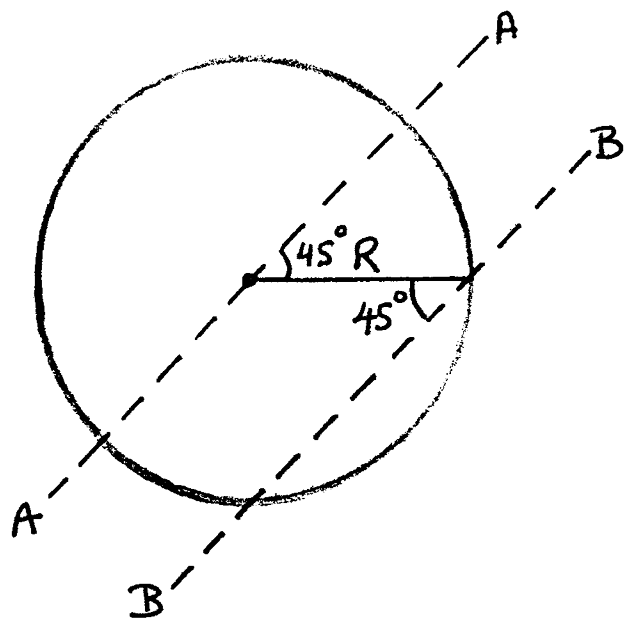
(ii) Let I_{OO} be the moment of inertia of the disc about an axis perpendicular to the disc and passing through the point O.

The moment of inertia of a disc about any diameter is the same. Therefore, by the perpendicular axis theorem,

$$I_{OO} = 2I_{AA} \; .$$

But from the table on p. 277, $I_{OO} = \frac{1}{2}MR^2$. Therefore

$$I_{AA} = \frac{1}{4}MR^2 \; .$$



Using the parallel axis theorem, we obtain

$$I_{BB} = I_{AA} + M\left(\frac{R}{\sqrt{2}}\right)^2 = \frac{1}{4}MR^2 + \frac{1}{2}MR^2 = \boxed{\frac{3}{4}MR^2 \; .}$$

We know that for rotations about a fixed axis

$$\tau = I\alpha \; ,$$

where α is the angular acceleration. Therefore in the two cases the angular accelerations are

$$\boxed{\frac{12\tau}{7Ma^2}} \quad \text{and} \quad \boxed{\frac{4\tau}{3MR^2} \; ,} \quad \text{respectively.}$$

Kinetic Energy and Rotational Motion about a Fixed Axis:

Problem 8 (Y&F): Two blocks, a pulley, and a tabletop with friction

9.85: In descending a distance *d*, gravity has done work $m_B gd$ and friction has done work $-\mu_K m_A gd$, and so the total kinetic energy of the system is $gd(m_B - \mu_K m_A)$. In terms of the speed *v* of the blocks, the kinetic energy is

$$K = \frac{1}{2}(m_A + m_B)v^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m_A + m_B + I/R^2)v^2,$$

where $\omega = v/R$, and condition that the rope not slip, have been used. Setting the kinetic energy equal to the work done and solving for the speed *v*,

$$v = \sqrt{\frac{2gd(m_B - \mu_k m_A)}{(m_A + m_B + I/R^2)}}.$$

Problem 9 (Y&F): Two disks pivoted on a rod

9.89: a) $\frac{1}{2}M_1R_1^2 + \frac{1}{2}M_2R_2^2 = \frac{1}{2}((0.80\,\text{kg})(2.50\times10^{-2}\,\text{m})^2 + (1.60\,\text{kg})(5.00\times10^{-2}\,\text{m})^2)$

$$= 2.25\times10^{-3}\,\text{kg}\cdot\text{m}^2.$$

b) See Example 9.9. In this case, $\omega = v/R_1$, and so the expression for *v* becomes

$$v = \sqrt{\frac{2gh}{1+(I/mR^2)}}$$

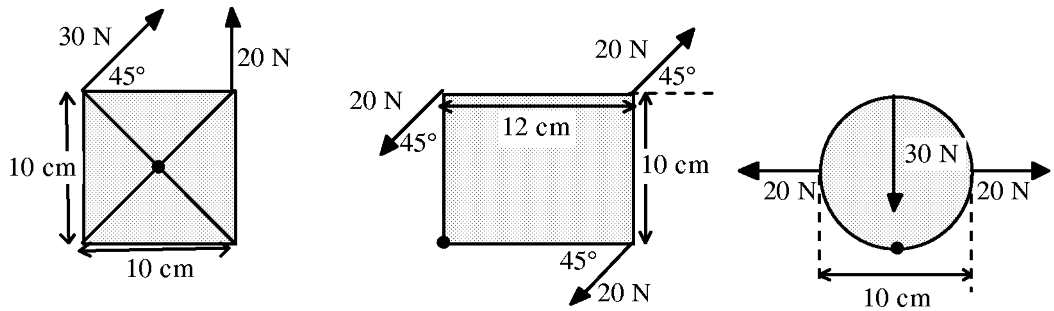
$$= \sqrt{\frac{2(9.80\,\text{m/s}^2)(2.00\,\text{m})}{(1+((2.25\times10^{-3}\,\text{kg}\cdot\text{m}^2)/(1.50\,\text{kg})(0.025\,\text{m})^2))}} = 3.40\,\text{m/s}.$$

c) The same calculation, using R_2 instead of R_1, gives $v = 4.95$ m/s, which is faster. This does make sense, because when the string is wrapped around the larger disk, the constraint equation $v = R\omega$ implies that one has a smaller ω for any given v. Thus less energy is used for the rotation of the disks, so more energy is available to allow the block to fall faster.

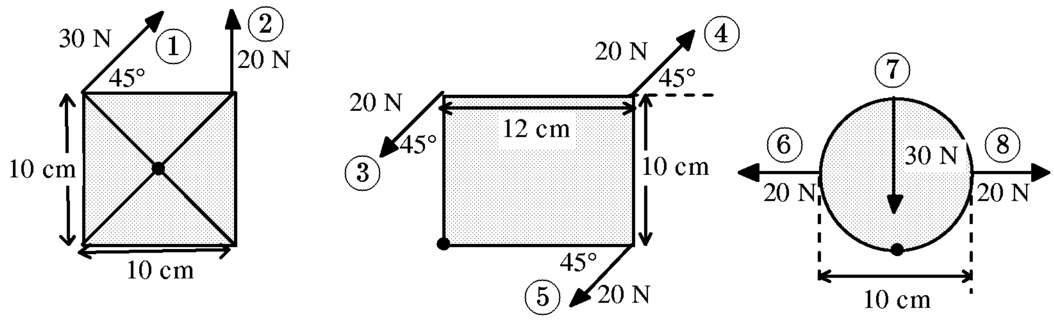
Torque and Angular Acceleration:

Problem 10 (SG): (H) Calculating torques about an axis

8B.2 (H) Calculate the net torque in each of the following cases about a fixed axis perpendicular to the page and passing through the indicated black dot. (Assume you are looking down on the xy-plane, so counterclockwise torques are positive.)



Answer: Numbering the forces for reference,



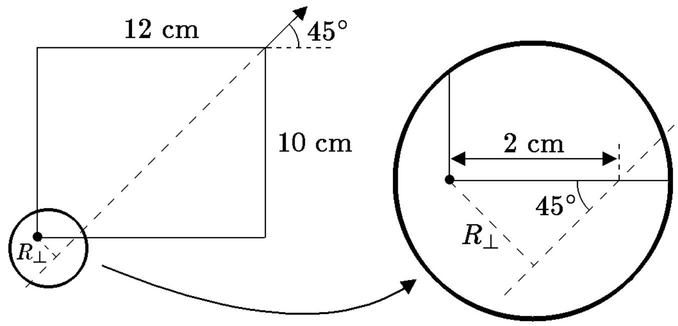
For two-dimensional rotation about a fixed axis, the torque can be calculated from

$$\tau = F_\perp R \quad \text{or} \quad \tau = \pm|\vec{\mathbf{F}}|R_\perp \ .$$

Here R is the distance from the point where the force is applied to the rotation axis, measured in the x-y plane. $F_\perp$ is the component of the force that is perpendicular to the radius vector, with positive defined as counterclockwise. $|\vec{\mathbf{F}}|$ is the magnitude of the force (assumed to be entirely in the x-y plane), and $R_\perp$ is the magnitude of the component of the radius vector that is perpendicular to the force. Geometrically, $R_\perp$ can be described as the distance, measured in the x-y plane, between the axis of rotation and the "line of action of the force," a line which goes through the point where the force is applied in the same direction as the force. The $\pm$ choice in the second formula is decided by the rule that torques that push in a counterclockwise direction are positive. The formulas are equivalent, so the choice of which to use is made for convenience.

Force 1: $F_\perp = -30$ N, $R = 5\sqrt{2}$ cm $\implies$ $\tau = -2.1$ N·m.
Force 2: $|\vec{\mathbf{F}}| = 20$ N, $R_\perp = 5$ cm, counterclockwise $\implies$ $\tau = 1.0$ N·m.
Object 1: total torque $\boxed{\tau = -1.1 \text{ N·m.}}$

Force 3: $F_\perp = 20/\sqrt{2}$ N, $R = 10$ cm $\implies$ $\tau = \sqrt{2}$ N·m.
Force 4: $|\vec{\mathbf{F}}| = 20$ N, $R_\perp = \sqrt{2}$ cm, counterclockwise $\implies$ $\tau = \sqrt{2}/5$ N·m, where the value of $R_\perp$ can be seen from the following diagram:

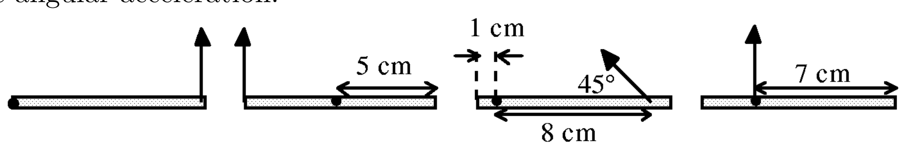


Force 5: $F_\perp = -20/\sqrt{2}$ N, $R = 12$ cm $\Longrightarrow$ $\tau = -1.2\sqrt{2}$ N·m.
Object 2: total torque $\boxed{\tau = 0}$

Force 6: $|\vec{\mathbf{F}}| = 20$ N, $R_\perp = 5$ cm, counterclockwise $\Longrightarrow$ $\tau = 1.0$ N·m.
Force 7: $|\vec{\mathbf{F}}| = 30$ N, $R_\perp = 0$ $\Longrightarrow$ $\tau = 0$.
Force 8: $|\vec{\mathbf{F}}| = 20$ N, $R_\perp = 5$ cm, clockwise $\Longrightarrow$ $\tau = -1.0$ N·m.
Object 3: total torque $\boxed{\tau = 0}$

Problem 11 (SG): Torque and angular acceleration of a pivoted rod

8E.2 Find the torque about the specified axis in each of the following situations, given that the force is 20 N in each case, the rod is 10 cm long, and the fixed axis is perpendicular to the page and passes through the black dot. If the mass of the rod is 0.5 kg, also find the angular acceleration.



Answer: The essential relationships needed for this problem are

$$\tau = F_\perp R \quad \text{and} \quad \alpha = \frac{\tau}{I} \ .$$

(i)

$$\boxed{\tau = 20 \text{ N} \times (.1 \text{ m}) = 2 \text{ N·m} \ , \ \text{counterclockwise.}}$$

$$I = \frac{1}{3}(0.5 \text{ kg})(.1 \text{ m})^2 = 0.0017 \text{ kg·m}^2 \ .$$

Therefore

$$\boxed{\alpha = 1200 \text{ rad/s}^2 \ , \ \text{counterclockwise.}}$$

(ii)

$$\boxed{\tau = 20 \text{ N} \times (0.05 \text{ m}) = 1 \text{ N·m} \ , \ \text{clockwise.}}$$

$$I = 2 \times \frac{1}{3}\left(\frac{0.5\ \text{kg}}{2}\right)(0.05\ \text{m})^2 = 4.2 \times 10^{-4}\ \text{kg}\cdot\text{m}^2\ .$$

Therefore

$$\boxed{\alpha = 2400\ \text{rad/s}^2\ ,\ \text{clockwise.}}$$

(iii)

$$\boxed{\tau = \left(20\ \text{N} \times \frac{1}{\sqrt{2}}\right) \times (0.08\ \text{m}) = 1.13\ \text{N}\cdot\text{m}\ ,\ \text{counterclockwise.}}$$

$$I = \frac{1}{3}\left(\frac{0.5\ \text{kg}}{10}\right)(0.01\ \text{m})^2 + \frac{1}{3}\left(0.5\ \text{kg} \times \frac{9}{10}\right)(0.09\ \text{m})^2 = 1.22 \times 10^{-3}\ \text{kg}\cdot\text{m}^2\ .$$

Therefore

$$\boxed{\alpha = 929\ \text{rad/s}^2\ ,\ \text{counterclockwise.}}$$

(iv)

$$\boxed{\tau = 0\ ,} \quad \text{and therefore} \quad \boxed{\alpha = 0\ .}$$

Problem 12 (Y&F): Frictional torque and a spinning spherical shell with point masses attached

10.7: $I = \frac{2}{3}MR^2 + 2mR^2$, where $M = 8.40\ \text{kg}$, $m = 2.00\ \text{kg}$

$I = 0.600\ \text{kg}\cdot\text{m}^2$

$\omega_0 = 75.0\ \text{rpm} = 7.854\ \text{rad/s}; \omega = 50.0\ \text{rpm} = 5.236\ \text{rad/s}; t = 30.0\ \text{s}, \alpha = ?$

$\omega = \omega_0 + \alpha t$ gives $\alpha = -0.08726\ \text{rad/s}^2$;

$\Sigma\tau = I\alpha,\ \tau_f = I\alpha = -0.0524\ \text{N}\cdot\text{m}$

Problem 13 (SG): A yo-yo unwinding under the force of gravity

10.10 A yo-yo consists of two solid disks connected by a central spindle. Each disk has mass $M = 20$ g and radius $R = 2.5$ cm; the central spindle has radius $r = 0.5$ cm and negligible mass. The yo-yo has 1 m of string attached; you may assume the string is of negligible mass and thin enough that it does not change the effective diameter of the spindle when it is wound up. If the yo-yo is released from rest, use energy conservation methods to deduce its angular velocity when it reaches the end of the string (assume the string was fully wound at the start and unwinds without slipping). If the string is firmly attached to the yo-yo, what do you expect it to do after reaching the end? What if the last piece of string is tied in a loop around the spindle rather than being firmly attached? Take $g = 10\ \text{m/s}^2$ and quote your results to two significant figures.

Draw a force diagram for the descending yo-yo, and hence calculate its acceleration. What effect would (a) decreasing the diameter of the central spindle; (b) decreasing the mass of the yo-yo; (c) decreasing the outer radius of the yo-yo have on its acceleration?

Answer: To make use of energy conservation, we will want to know the moment of inertia of the yo-yo for rotation about the axis that goes through the center of each disk, so that we can calculate its rotational kinetic energy. The moment of inertia of a uniform disk about its own axis is given on the table of p. 277 of the *Study Guide*, remembering that a "disk" is actually a uniform solid cylinder. It is given by $I_{\text{disk}} = \frac{1}{2}mR^2$. Since the yo-yo consists of two disks of mass M and radius R, its moment of inertia is $I_{\text{yo-yo}} = MR^2$. Since the central spindle has negligible mass, its contribution to the moment of inertia is negligible.

If we take y as the vertical coordinate, we can choose the zero of potential energy so that $U = 2Mgy$, since the mass of the yo-yo is $2M$. If we take $y = 0$ to be the initial height of the yo-yo, then the initial energy of the yo-yo is zero:

$$E_{\text{tot}} = E_k + U = 0 + 0 = 0 \ .$$

As the yo-yo descends the string unwinds, so the rotational motion of the yo-yo is linked to the vertical motion by a rolling constraint

$$v_y = r\omega \ ,$$

where r is the radius of the spindle around which the string is wound. (If you don't see this, consider what happens when the yo-yo goes through one full rotation. The angle of one full rotation is $\Delta\theta = 2\pi$, and the length of string that is released is the length of one turn of string around the spindle, which is the circumference of the spindle, $\Delta\ell = 2\pi r$. Thus, for one full rotation, $\Delta\ell = r\,\Delta\theta$. But $\Delta\theta$ and $\Delta\ell$ are both proportional to the number of rotations, so the relation $\Delta\ell = r\,\Delta\theta$ holds for ANY number of rotations, even fractional numbers. Dividing this equation by Δt, and taking the limit $\Delta t \to 0$, we get $v_y = r\omega$.) The energy of the yo-yo after falling a distance ℓ is then

$$E(\ell) = \frac{1}{2}(2M)v_y^2 + \frac{1}{2}I_{\text{yo-yo}}\omega^2 - 2Mg\ell = Mr^2\omega^2 + \frac{1}{2}MR^2\omega^2 - 2Mg\ell \ .$$

By conservation of mechanical energy this number must be zero, so

$$\boxed{\omega = \sqrt{\frac{4g\ell}{R^2 + 2r^2}} \ .}$$

Numerically,

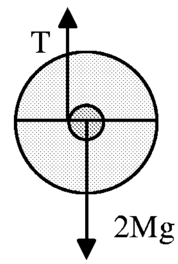
$$\omega = \sqrt{\frac{4(10\ \text{m/s}^2)\,(1\ \text{m})}{(0.025\ \text{m})^2 + 2(0.005\ \text{m})^2}} = \boxed{250\ \text{s}^{-1} \ .}$$

When the yo-yo reaches the end of the string, if it is firmly attached, the rotation of the yo-yo will cause the string to wind again around the spindle, pulling the yo-yo back up. In the absence of friction, the yo-yo will rise to the same height from which it started. If the string instead is tied in a loop so that it can slip without friction, then the yo-yo will spin freely when it reaches the bottom, without rewinding the string.

On the other hand, even if the string is wound in a loop, if there is enough friction it can cause the string to rewind so that the yo-yo comes back up. If the yo-yo is spinning freely at the bottom of its string, sometimes a tug on the string can increase the normal force of the string on the spindle so that the force of friction increases, and the yo-yo can start to move upward.

A force diagram for the yo-yo is shown at the right. The equation for the y-motion is

$$T - 2Mg = 2Ma_y \ ,$$



and the equation for the rotation about the center of mass is

$$\boldsymbol{\tau} = -Tr = I_{\text{yo-yo}}\alpha \ ,$$

where I have defined counterclockwise motion as positive. Differentiation of the constraint equation $v_y = r\omega$ implies that

$$a_y = r\alpha \ ,$$

so these three equations provide enough information to solve for the three unknowns T, a_y, and α. Solving the 3rd equation for α and inserting into the second, one finds

$$-Tr^2 = I_{\text{yo-yo}}a_y = MR^2 a_y \ .$$

One can solve this equation for T and insert it into the first, giving

$$\boxed{a_y = -\frac{g}{1 + \frac{R^2}{2r^2}} \ .}$$