MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

PROBLEM SET 9 SOLUTIONS

April 7, 2005

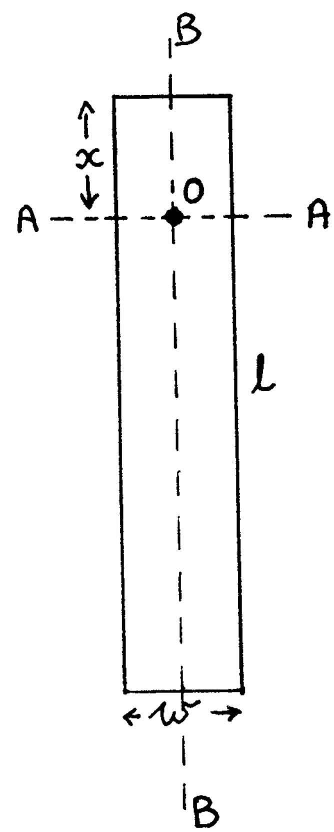
Angular Momentum in Two Dimensions:

Problem 1 (SG): Rotation of a ruler about an off-center hole.

8D.2 (H) A ruler of mass m, length ℓ and width w has a hole drilled in it a short distance x from one end and equidistant from both sides. It is anchored on a frictionless air table by a nail driven through the hole and is then set rotating about the nail such that its far end (a distance $\ell - x$ from the hole) is moving with (linear) speed v. Obtain expressions for the rotational kinetic energy K and angular momentum L of the ruler with respect to the axis through the nail. Calculate the values of K and L if the ruler is 35 cm long, 2 cm wide and has a mass of 50 g, the hole is 1 cm from one end, and the other end is moving at 0.2 m/s.

Answer:

Let I_{OO} be the moment of inertia of the ruler about the nail. Using the axes shown on the diagram, the perpendicular axis theorem implies



$$I_{\rm OO} = I_{\rm AA} + I_{\rm BB} \ .$$

From the second item in the table on p. 277, we find that

$$I_{\rm AA} = \frac{1}{3} m_1 x^2 + \frac{1}{3} m_2 (\ell - x)^2 \ ,$$

where

$$m_1 = \frac{x}{\ell} m \ , \qquad m_2 = \frac{\ell - x}{\ell} m \ ,$$

and that

$$I_{\rm BB} = 2 \times \frac{1}{3} \left(\frac{m}{2}\right) \left(\frac{w}{2}\right)^2 \ .$$

Therefore

$$I_{\rm OO} = \frac{1}{3} \frac{x}{\ell} m x^2 + \frac{1}{3} \frac{\ell - x}{\ell} m (\ell - x)^2 + \frac{m w^2}{12} \ .$$

Simplifying, we find

$$I_{\rm OO} = m \left[\frac{x^3}{3\ell} + \frac{(\ell - x)^3}{3\ell} + \frac{w^2}{12} \right] = m \left[\frac{w^2}{12} + \frac{\ell^2}{3} - \ell x + x^2 \right] \ .$$

The speed v of the far end of the ruler is related to the angular velocity ω by $v = (\ell - x)\omega$, so $\omega = v/(\ell - x)$. Then

$$K = \frac{1}{2} I_{OO}\omega^2 = \boxed{\frac{1}{2}m\left[\frac{w^2}{12} + \frac{\ell^2}{3} - \ell x + x^2\right]\frac{v^2}{(\ell - x)^2}\ .}$$

The angular momentum is

$$L = I_{OO}\omega = \boxed{m\left[\frac{w^2}{12} + \frac{\ell^2}{3} - \ell x + x^2\right]\frac{v}{\ell - x}\ .}$$

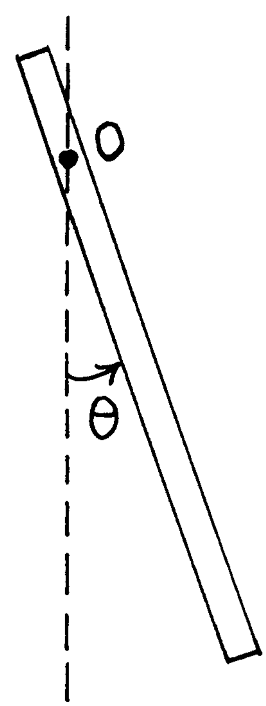
Using the numbers given, $I_{OO} = 0.0019\ \text{kg}\cdot\text{m}^2$, and

$$\boxed{K = 3.3 \times 10^{-4}\ \text{J}; \qquad L = 1.1 \times 10^{-3}\ \text{kg}\cdot\text{m}^2/\text{s}\ .}$$

8E.4 (H) The ruler of problem 8D.2, with length ℓ, width w and mass m, is suspended by a rod passing through the hole x from one end, which thus forms a fixed axis perpendicular to the plane of the ruler. If the ruler is held so that it makes an angle of θ to the vertical and then released, what is the torque acting on it, and what is its angular acceleration?

Answer:

Define θ to be positive as shown on the diagram, i.e we consider the axis to be going out of the plane of the paper through point O. When the ruler is displaced as shown, there will be a torque $\boldsymbol{\tau}$ acting on the ruler (about the given axis) given by:



$$\boxed{\boldsymbol{\tau} = -Mg\left[\frac{\ell}{2} - x\right]\sin\theta\ .}$$

Now

$$\boldsymbol{\tau} = I\alpha = I\frac{d^2\theta}{dt^2}\ ,$$

and

$$I = M\left[\frac{w^2}{12} + \frac{\ell^2}{3} - \ell x + x^2\right]$$

(see solution to problem 8D.2).

Therefore

$$I\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = -Mg\left[\frac{\ell}{2} - x\right]\sin\theta \ ,$$

or

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}t^2} = \frac{-Mg\left[\frac{\ell}{2} - x\right]\sin\theta}{M\left[\frac{w^2}{12} + \frac{\ell^2}{3} - \ell x + x^2\right]} = \boxed{-\omega_0^2\,\sin\theta\ ,}$$

where

$$\boxed{\omega_0 = \sqrt{\frac{g\left[\frac{\ell}{2} - x\right]}{\left[\frac{w^2}{12} + \frac{\ell^2}{3} - \ell x + x^2\right]}}\ .}$$

Problem 2 (Y&F): More things that you can do when your frictionless table has a hole in it

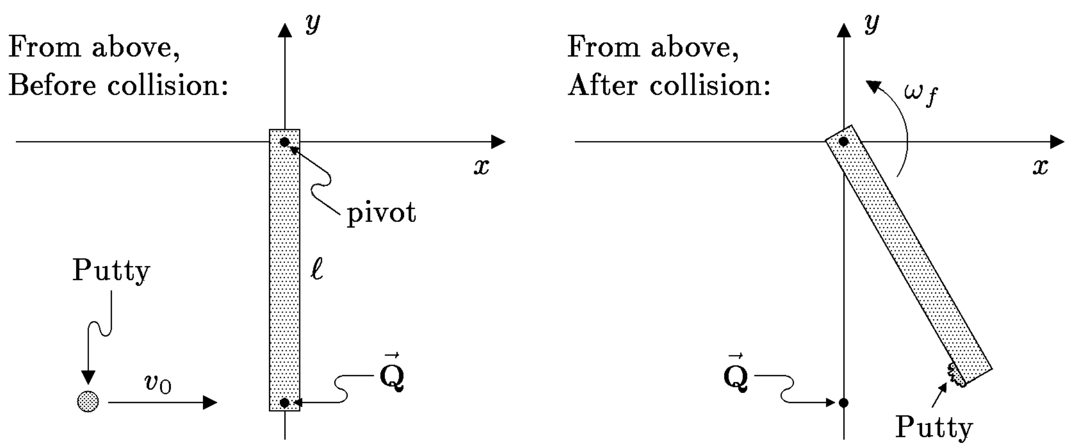
10.39: a) The net force is due to the tension in the rope, which always acts in the radial direction, so the angular momentum with respect to the hole is constant.

b) $L_1 = m\omega_1 r^2{}_1, L_2 = m\omega_2 r_2{}^2$, and with $L_1 = L_2, \omega_2 = \omega_1(r_1/r_2)^2 = 7.00\ \mathrm{rad}/s$.

c) $\Delta K = (1/2)m((\omega_2 r_2)^2 - (\omega_1 r_1)^2) = 1.03\times10^{-2}$ J.

d) No other force does work, so 1.03×10^{-2} J of work were done in pulling the cord.

Problem 3 (SG) *STUDY:* A ball of putty hitting a pivoted stick. Assume that the pivot is completely frictionless.



10.1 (S) On a frictionless horizontal table a slender rigid rod of mass M and length ℓ is attached at one end to a fixed pivot. A nonrotating disk of putty, with mass m, is moving with speed v_0 perpendicular to the line of the rod, and collides with it at point Q, at the opposite end from the pivot. The disk sticks to the end of the rod, so that after the collision they move off together. Assume that the disk is small enough to be treated as a point particle.

(a) Just before the collision, what is the angular momentum of the system of rod plus disk (i) about the the z-axis, (ii) about an axis perpendicular to the page, through the point Q?

(b) Just after the collision, what is the angular velocity of the rod/disk system about the pivot? What is the system's angular momentum about the axis perpendicular to the page and through the point Q? Explain.

(c) Now suppose that the disk is uniform and has a small radius R. It is initially rotating counterclockwise (as seen from above) with an angular speed ω_0 so large that the resulting internal angular momentum cannot be neglected (although R is still negligible compared to ℓ). If the disk's translational speed when it hits the rod is the same as before, what is the final angular velocity about the pivot in this case?

Answer: See complete solution in the *Study Guide*.

Rotation about a Moving Axis— Rolling:

Problem 4 (SG) *STUDY:* Rolling without slipping

8E.8 (S) A rigid body of mass M and having a circular cross-section of radius R rolls without slipping down a slope making an angle θ to the horizontal. The moment of inertia of the body about its central axis of symmetry is kMR^2, where k is a numerical constant.

(a) What is its speed when it has descended through a vertical distance h?

(b) What is the minimum coefficient of static friction required to ensure that it rolls, rather than slides, down the slope?

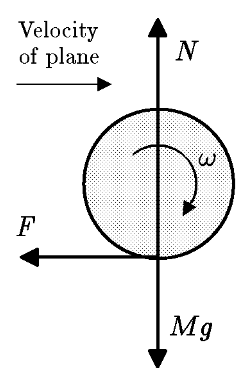
Answer: See complete solution in the *Study Guide*.

Problem 5 (SG): Spinning wheels on an aircraft as it lands

9D.6 (H) An aircraft lands at a speed v_0. Before it touches down, its wheels are not rotating. Describe in words what happens when the wheels touch the ground. Assuming that each wheel has radius R and moment of inertia I and supports a weight Mg, and that the pilot does not apply reverse thrust until the aircraft is no longer skidding, how fast is the plane moving when it stops skidding?

Answer: When the wheels touch the ground they do not rotate and hence are skidding. Kinetic friction between the wheel and the ground will imply a torque that accelerates the rotation of the wheel until it stops skidding. The same friction force will decelerate the translational motion of the plane. The forces on each wheel are gravity Mg, a compensating normal force $N = Mg$, and a backward kinetic friction force $F = \mu_k N = \mu_k Mg$. The friction force causes a uniform deceleration $a = F/M = \mu_k g$ of the plane, so that after a time t its linear velocity is reduced to

$$v(t) = v_0 - \mu_k g t \ .$$



Since motion to the right, which we take as positive, corresponds to clockwise rotation of the wheel, for purposes of this problem we will drop the usual convention that counterclockwise is positive, and instead define clockwise as the positive direction for ω, α, and $\boldsymbol{\tau}$. With this convention, the friction force exerts a positive torque $\boldsymbol{\tau} = FR = \mu_k MgR$ about the center of mass axis (the axle) of the wheel, which causes an angular acceleration

$$\alpha = \frac{\boldsymbol{\tau}}{I} = \frac{\mu_k MgR}{I} \ .$$

Hence, the resulting angular velocity of the wheels is

$$\omega(t) = \alpha t = \frac{\mu_k MgRt}{I} \ .$$

The wheels stop skidding when the rolling condition $v(t) = R\omega(t)$ is satisfied, i.e. when

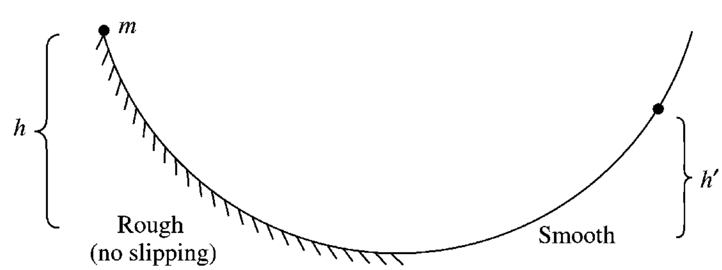
$$v_0 - \mu_k gt = \frac{\mu_k MgR^2 t}{I} \quad \Longrightarrow \quad \mu_k gt = \frac{v_0}{1 + MR^2/I} \ .$$

The velocity at that time t is given by

$$v(t) = v_0 - \frac{v_0}{1 + MR^2/I} = \frac{MR^2}{MR^2 + I} v_0 \ .$$

Problem 6 (Y&F): Marble rolling in a bowl, with and without slipping

10.24:



$v_{cm} = R\omega$ for no slipping

a) Get v at bottom:

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2$$

$$mgh = \frac{1}{2}mv^2 + \frac{1}{2}\left(\frac{2}{5}mR^2\right)\left(\frac{v}{R}\right)^2$$

$$v = \sqrt{\frac{10}{7}gh}$$

Now use energy conservation. Rotational *KE* does not change

$$\frac{1}{2}mv^2 + KE_{\text{Rot}} = mgh' + KE_{\text{Rot}}$$

$$h' = \frac{v^2}{2g} = \frac{\frac{10}{7}gh}{2g} = \frac{5}{7}h$$

(b) $mgh = mgh' \rightarrow h' = h$ With friction on both halves, all the *PE* gets converted back to *PE*. With one smooth side, some of the *PE* remains as rotational *KE*.

Problem 7 (SG) ***STUDY:*** **Pulling a yo-yo along the floor by its string**

10.11 (S) A yo-yo consists of two solid disks, each of mass M and radius R, connected by a central spindle of radius r and negligible mass. A light string is coiled around the central spindle; we shall assume that its thickness is sufficiently small that winding or unwinding it has no effect on the effective radius of the spindle. The yo-yo is placed upright on a flat surface and the string is pulled gently at an angle θ to the horizontal. What happens? Assume that the surface is rough enough that the yo-yo rolls without slipping.

Answer: See complete solution in the *Study Guide*.

Problem 8 (Y&F): The center of percussion of a baseball bat

10.98: The velocity of the center of mass will change by $\Delta v_{\text{cm}} = \frac{J}{m}$, and the angular velocity will change by $\Delta\omega = \frac{J(x - x_{\text{cm}})}{I}$. The change is velocity of the end of the bat will then be $\Delta v_{\text{end}} = \Delta v_{\text{cm}} - \Delta\omega x_{\text{cm}} = \frac{J}{m} - \frac{J(x - x_{cm})x_{\text{cm}}}{I}$. Setting $\Delta v_{\text{end}} = 0$ allows cancellation of J, and gives $I = (x - x_{\text{cm}})x_{\text{cm}}m$, which when solved for x is

$$x = \frac{I}{x_{\text{cm}}m} + x_{\text{cm}} = \frac{(5.30\times10^{-2}\,\text{kg}\cdot\text{m}^2)}{(0.600\text{ m})(0.800\text{ kg})} + (0.600\text{ m}) = 0.710\text{ m}.$$

Energy, Power, and Rotations:

Problem 9 (Y&F): A child's work in pushing a merry-go-round

10.27: a) $\omega = \alpha\Delta t = (FR/I)\Delta t = ((18.0\ \text{N})(2.40\ \text{m})/(2100\ \text{kg}\cdot\text{m}^2))(15.0\ \text{s}) = 0.3086\ \text{rad/s}$, or 0.309 rad/s to three figures.

b) $W = K_2 = (1/2)I\omega^2 = (1/2)\times\ (2.00\ \text{kg}\cdot\text{m}^2)(0.3086\ \text{rad/s})^2 = 100\ \text{J}$.

c) From either $P = \tau\omega_{\text{ave}}$ or $P = W/\Delta t$, $P = 6.67$ W.

Problem 10 (Y&F): Energy and power for a wheel undergoing uniform angular acceleration

10.55: a) $P = \tau\omega = \tau\alpha t = \tau\left(\frac{\tau}{I}\right)t = \tau^2\left(\frac{t}{I}\right)$.

b) From the result of part (a), the power is $(500\ \text{W})\left(\frac{60.0}{20.0}\right)^2 = 4.50\ \text{kW}$.

c) $P = \tau\omega = \tau\sqrt{2\alpha\theta} = \tau\sqrt{2(\tau/I)\theta} = \tau^{3/2}\sqrt{2\theta/I}$.

d) From the result of part (c), the power is $(500\ \text{W})\left(\frac{6.00}{20.00}\right)^{3/2} = 2.6\ \text{kW}$.

e) No, there is no contradiction. Since $\theta = \frac{1}{2}(\tau/I)t^2$, the equation in part (a) is clearly consistent with the equation in part (c). In words, the point is that both equations tell you what would happen if you changed the torque τ and measured the power P, but they still refer to different experiments. The equation in (a) tells you that $P \propto \tau^2$ if you made all your measurements at a fixed time after the wheel started to rotate. On the other hand, the equation in (c) tells you that you would find $P \propto \tau^{3/2}$ if you made all your measurements after the wheel had rotated through some fixed angle.