MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

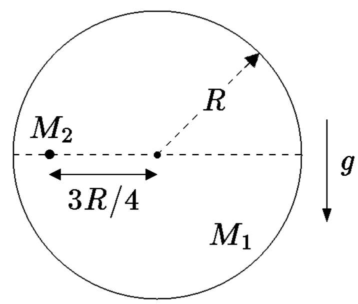
PROBLEM SET 10 SOLUTIONS

April 14, 2005

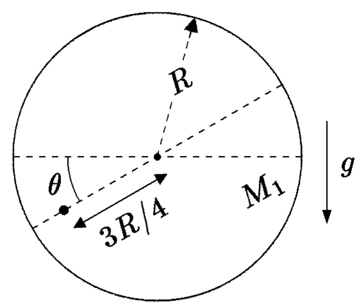
Rotation About a Fixed Axis:

Problem 1 (Quiz 8, Problem 3, slightly revised): A pivoted disk with attached weights

A uniform disk of mass M_1 and radius R is pivoted on a frictionless horizontal axle through its center. A small marble of mass M_2 is attached to the disk at radius $3R/4$, at the same height as the axle. Assume that the marble is small enough to be treated as a point mass. The acceleration of gravity is downward, with magnitude g.



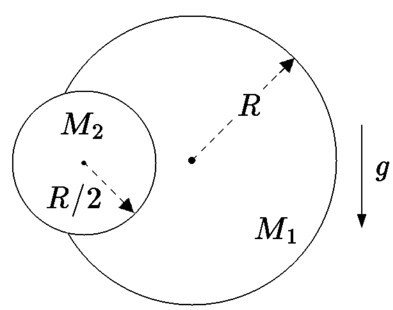
(a) If this system is released from rest to rotate about the pivot, what will be the angular acceleration α_0 of the disk immediately after it is released?



(b) After the disk has rotated through an angle θ, what will be the angular acceleration α_1?

(c) What will be the maximum angular velocity $\omega_{\max}$ that the disk will reach in its subsequent motion?

(d) Now consider the situation if the marble is replaced by a disk of radius $R/2$, with the same mass M_2, located with its center at the same place where the marble was located in part (a). (When calculating the torque on this disk, you can use the fact that the torque caused by gravity can be calculated as if the force of gravity were a single force acting at the center of mass of the object.) For this case, find the angular acceleration α_0' immediately after the system is released from rest.



(e) For the case described in part (d), what will be the maximum angular velocity $\omega_{\max}'$ that the disk will reach in its subsequent motion?

Answer:

(a) Since the axle goes through the center of mass of the disk of mass M_1, the gravitational force on this disk does not result in any torque about the axle. But there is a torque caused by the gravitational force on M_2, given by

$$\tau = R_\perp F = \frac{3}{4} M_2 g R \ .$$

The moment of inertia of the combined system about the axle is that of the disk M_1 plus the mass M_2, so

$$I = \frac{1}{2} M_1 R^2 + M_2 \left(\frac{3R}{4}\right)^2 = \frac{1}{16}(8M_1 + 9M_2)R^2 \ ,$$

where the moment of inertia of the disk is taken directly from the table in the formula sheets. The angular acceleration immediately after release is therefore

$$\boxed{\alpha = \frac{\tau}{I} = \frac{12 M_2 g}{(8M_1 + 9M_2)R} \ .}$$

(b) The only change is in the calculation of the torque, which in general is given by $\tau = R_\perp F$, where $R_\perp$ is the component of the radius vector perpendicular to the force. In this case $R_\perp$ is the horizontal component of the vector from the pivot to the marble, which is $(3R/4)\cos\theta$. Thus α is modified from the answer to part (a) by the insertion of this factor of $\cos\theta$:

$$\boxed{\alpha = \frac{12 M_2 g \cos\theta}{(8M_1 + 9M_2)R} \ .}$$

(c) The maximum angular velocity will be attained when M_2 is at the bottom of its motion. The value of the angular velocity can be determined by using the conservation of energy. The potential energy of the disk M_1 does not change, since its center of mass does not move, so the only potential energy that needs to be considered is that of M_2. This potential energy can be written $U = M_2 g y$, where y is the vertical coordinate, measured from an arbitrary origin. I will take that origin as the height of the axle. Thus $U_{\text{initial}} = 0$, and U_{final} (at the bottom of the motion) is $-M_2 g(3R/4)$. Then

$$E_{\text{initial}} = 0$$

$$E_{\text{final}} = \frac{1}{2} I \omega_f^2 - \frac{3}{4} M_2 g R$$

$$E_{\text{final}} = E_{\text{initial}} \quad \Longrightarrow \quad \boxed{\omega_f = \sqrt{\frac{3 M_2 g R}{2I}} = \sqrt{\frac{24 M_2 g}{(8M_1 + 9M_2)R}} \ .}$$

(d) The only difference between this case and the previous one is the moment of inertia of the disk of mass M_2. According to the table, the moment of inertia of this disk about its own center is $\frac{1}{2}M_2(R/2)^2$. But we need the moment of inertia about the center of the larger disk, for which we have to use the parallel axis theorem:

$$I_{\parallel} = I_{\rm cm} + Md^2 = \frac{1}{2}M_2\left(\frac{R}{2}\right)^2 + M_2\left(\frac{3R}{4}\right)^2 = \frac{11}{16}M_2R^2 \ .$$

So in this case the total moment of inertia is given by

$$I' = \frac{1}{2}M_1R^2 + \frac{11}{16}M_2R^2 = \frac{1}{16}(8M_1 + 11M_2)R^2 \ .$$

The torque is the same as in part (a), since the torque due to the gravitational force on M_2 can be calculated as if the entire force acted on the center of mass. Thus,

$$\boxed{\alpha'_0 = \frac{\tau}{I'} = \frac{12M_2g}{(8M_1 + 11M_2)R} \ .}$$

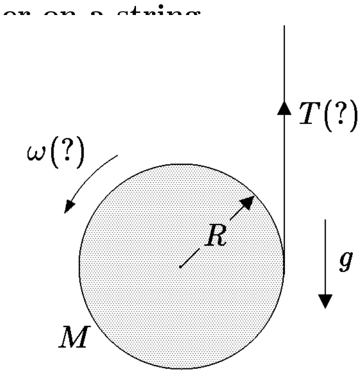
(e) The maximum angular velocity is also the same as it was in the previous case, except for the modification of the moment of inertia.

$$\boxed{\omega'_f = \sqrt{\frac{3M_2gR}{2I'}} = \sqrt{\frac{24M_2g}{(8M_1 + 11M_2)R}} \ .}$$

Rotation about a Moving Axis:

Problem 2: (Quiz 9, Problem 3, slightly revised): A cylinder on a string

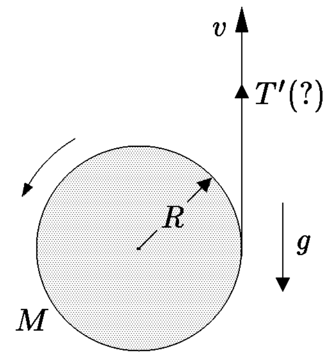
A solid cylinder of radius R and mass M is wrapped with an inextensible string of negligible mass. One end of the string is tied to the ceiling, and the cylinder is allowed to fall with its axis horizontal, as the string unrolls. Take the acceleration of gravity as g, downward, with $g > 0$.



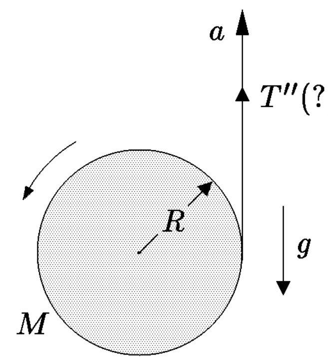
(a) Find the angular velocity ω of the cylinder after it falls a distance ℓ, starting from rest with the string taut.

(b) Find the tension T in the string, as the cylinder is falling.

(c) Now suppose that instead of the string being tied to the ceiling, it is being held by a person who pulls the end upward with a constant speed v. What is the tension T' in the string in this case?



(d) Now suppose that instead of pulling the string upward with a constant speed, the person pulls the end of the string upward with a constant acceleration a. What is the tension T'' of the string in this case?



Answer:

(a) This can be solved most easily by using conservation of energy. If we define the gravitational potential energy to be zero at the bottom of the fall, then the initial energy is all potential energy,

$$E_{\text{initial}} = U(\ell) = Mg\ell \; .$$

After falling a distance ℓ the cylinder will be moving at speed v and angular velocity ω, so its energy will be

$$F_{\text{final}} = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2 \; .$$

The string is being unwound from a circle of radius R, so v will be related to ω by $v = R\omega$. The moment of inertia for the solid cylinder is given in the table as $I = \frac{1}{2}MR^2$ So, setting the initial and final energies equal,

$$Mg\ell = \frac{1}{2}MR^2\omega^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2 = \frac{3}{4}MR^2\omega^2 \; ,$$

so

$$\boxed{\omega = \frac{2}{R}\sqrt{\frac{g\ell}{3}} \; .}$$

(b) METHOD 1: The tension can be found from the previous answer, adding a little information. The final vertical velocity v_y, after falling the distance ℓ, is given by

$v_y = -R\omega = -2\sqrt{g\ell/3}$. Here I am defining v_y as being positive when upward, so its value for this problem is negative. For uniform acceleration we know that $v^2 = 2a\ell$, so the cylinder is undergoing uniform acceleration with $a_y = -2g/3$. The $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ equation for motion in the vertical (y) direction is $F_y = T - Mg = Ma_y$, so $a_y = -2g/3$ implies that

$$\boxed{T = \frac{1}{3}Mg\ .}$$

METHOD 2: Alternatively, we can use the torque and force equations to find the tension directly. Once again, the $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ equation for vertical motion is

$$F_y = T - Mg = Ma_y\ .$$

The torque is given by $\boldsymbol{\tau} = TR$, and it is positive (counterclockwise). So

$$\boldsymbol{\tau} = TR = I\alpha\ .$$

The rolling constraint for these sign conventions is $v_y = -R\omega$, since a positive (counterclockwise) rotation corresponds to motion of the cylinder downward (negative v_y). Differentiation of the rolling constraint implies that $a_y = -R\alpha$, so the torque equation can be rewritten as

$$TR = -Ia_y/R = -\frac{1}{2}MRa_y\ ,$$

where I used $I = \frac{1}{2}MR^2$. The torque equation then becomes $T = -\frac{1}{2}Ma_y$, which can be substituted into the first equation to give

$$T - Mg = -2T\ ,$$

leading immediately to $\boxed{T = \frac{1}{3}Mg\ .}$

(c) The answer is the same as part (b), $\boxed{T' = \frac{1}{3}Mg\ .}$ One way to see this is to work the problem in the frame of reference that is moving upwards at speed v relative to the ground, and then the problem becomes identical to the previous one. If the problem is attacked in the frame of reference of the ground, then one can use Method 2 above, and the only change is in the rolling constraint. Now if ω were zero the cylinder would be moving upward with speed v, so the rolling constraint becomes

$$v_y = -R\omega + v\ .$$

Since v is a constant, differentiation gives $a_y = -R\alpha$ as before, so the derivation goes through without any change.

(d) This time the rolling constraint can be written as

$$v_y = -R\omega + v(t)\ ,$$

where $v(t)$ is the instantaneous velocity of the hand that is pulling the string upwards. Now when we differentiate with respect to time,

$$a_y = -R\alpha + a \ ,$$

where $\mathrm{d}v/\mathrm{d}t = a$, where a is the upward acceleration of the person's hand, as specified in the problem. We still have

$$T''R = I\alpha \ ,$$

so

$$T''R = -I(a_y - a)/R = -\frac{1}{2}MR(a_y - a) \ .$$

Then $Ma_y = Ma - 2T''$, which can be inserted into the $\vec{\mathbf{F}} = M\vec{\mathbf{a}}$ equation to give

$$T'' - Mg = Ma - 2T'' \ ,$$

so

$$\boxed{T'' = \frac{1}{3}M(g + a) \ .}$$

Problem 3: Striking a billiard ball with a cue

A billiard ball, which we treat as a solid ball of mass M and radius R, is at rest on a frictionless table. The ball is hit sharply by a cue, which imparts a horizontal impulse of momentum at a height b above the center of the ball. The strike is centered, which means that the impulse of momentum lies in the vertical plane that includes the center of the sphere and the point of impact.

Even though the surface is frictionless, if the ball is hit at just the right height, it will roll without slipping immediately after it is hit. For what value of b will this happen?

Answer:

The impulsive momentum will result in an impulsive torque about the center of mass, since

$$\frac{\mathrm{d}\vec{\mathbf{L}}}{\mathrm{d}t} = \vec{\mathbf{r}} \times \vec{\mathbf{F}}(t)$$

can be integrated over time to give

$$\vec{\mathbf{L}}_{\text{final}} - \vec{\mathbf{L}}_{\text{initial}} = \int \vec{\mathbf{r}} \times \vec{\mathbf{F}}(t)\,\mathrm{d}t = \vec{\mathbf{r}} \times \int \vec{\mathbf{F}}(t)\,\mathrm{d}t = \vec{\mathbf{r}} \times \vec{\mathbf{J}} \ ,$$

where

$$\vec{\mathbf{J}} = \int \vec{\mathbf{F}}(t)\,\mathrm{d}t$$

is the impulse, and $\vec{\mathbf{r}}$ is a vector from the center of the billiard ball to the point where the force is applied. Since in this case $\vec{\mathbf{L}}_{\text{initial}} = 0$, then

$$\vec{\mathbf{L}}_{\text{final}} = \vec{\mathbf{r}} \times \vec{\mathbf{J}} \ .$$

Then

$$\left|\vec{\mathbf{L}}_{\text{final}}\right| = I\omega = bJ\ ,$$

where $J \equiv \left|\vec{\mathbf{J}}\right|$, and the term bJ arises because $r\sin\theta = b$, where θ is the angle between $\vec{\mathbf{r}}$ and $\vec{\mathbf{J}}$, which is the same as the angle between $\vec{\mathbf{r}}$ and the horizontal.

The condition for rolling without slipping is $v = R\omega$, so this becomes

$$v = \frac{bRJ}{I}\ .$$

But we also know that the momentum of the ball after being hit is $\vec{\mathbf{J}}$, so its speed is $v = J/M$. Setting the two expressions for v equal to each other,

$$\frac{bRJ}{I} = \frac{J}{M} \quad \Longrightarrow \quad b = \frac{I}{MR}\ .$$

For a sphere, $I = \frac{2}{5}MR^2$, so

$$\boxed{b = \frac{2}{5}R\ .}$$

Rotation, Angular Momentum, and Torque in Three Dimensions:

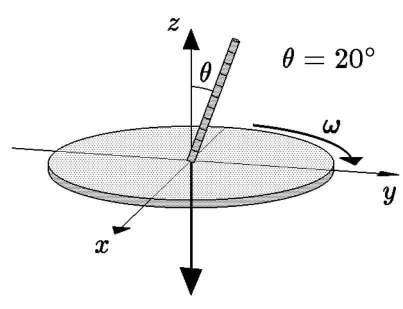
Problem 4 (SG) *STUDY:* Verifying that torque is the rate of change of angular momentum

9C.2 (S) Use the component form of the vector product to verify that $\vec{\boldsymbol{\tau}} = \dfrac{d\vec{\mathbf{L}}}{dt}$.

Answer: See complete solution in the *Study Guide*.

Problem 5 (SG) *STUDY:* A skewed rod on a turntable

9D.1 (S) A uniform rod is fixed to a rotating horizontal turntable so that its lower end is on the axis of the turntable and it makes an angle of 20° to the vertical. (It is thus rotating with uniform angular velocity about an axis passing through one end and inclined at 20° to the direction of the rod.) If the turntable is rotating clockwise as seen from above, what is the direction of the rod's angular velocity vector?



(a) vertically downwards; (b) down at 20° to the vertical;
(c) up at 20° to the vertical; (d) vertically upwards.

What is the direction of its angular momentum vector (calculated about its lower end)?

(a) vertically downwards; (b) down at 20° to the horizontal;
(c) up at 20° to the horizontal; (d) vertically upwards.

Is there a torque acting on it, and if so in what direction?

(a) yes, vertically; (b) yes, horizontally;
(c) yes, at 20° to the horizontal; (d) no.

Answer: See complete solution in the *Study Guide*.

Problem 6 (SG): Rotating meteoroid

9D.9 A meteoroid in empty space rotates about its center of mass. We choose a coordinate system in which the center of mass is at rest at the origin, and the angular velocity at the time of interest points along the z-axis, so that $\vec{\boldsymbol{\omega}} = [0, 0, \omega_z]$. A small particle, or *chondrule*, of the mineral olivine is embedded in the meteroid at position $\vec{\mathbf{r}}_0 = [x_0, 0, z_0]$. The chondrule has mass M. (Note that the chondrule is part of the meteoroid, so its mass and position have been included in the calculation of the meteoroid's center of mass.)

(a) What are the components of the velocity vector $\vec{\mathbf{v}}$ of the chondrule?

(b) What is the angular momentum of the chondrule (i) about the z-axis, i.e. the instantaneous axis of rotation; (ii) about the center of mass of the meteoroid?

(c) The overall angular momentum vector of the meteoroid about its center of mass makes an angle θ with the angular velocity vector. Describe in words the motion of the meteoroid as seen by a passing space probe.

Answer:

(a) The velocity is given by $\vec{\mathbf{v}} = \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}$, so

$$\vec{\mathbf{v}} = [0, 0, \omega_z] \times [x_0, 0, z_0] \ .$$

Using the component formula for the cross product $\vec{\mathbf{c}} = \vec{\mathbf{a}} \times \vec{\mathbf{b}}$,

$$\begin{aligned} c_x &= a_y b_z - a_z b_y \\ c_y &= a_z b_x - a_x b_z \\ c_z &= a_x b_y - a_y b_x \ , \end{aligned}$$

one finds

$$\boxed{\vec{\mathbf{v}} = [0, \omega_z \, x_0, 0] \ .}$$

(b) The angular momentum about the z-axis is defined by $L = I\omega$, where $\omega = |\vec{\boldsymbol{\omega}}| = \omega_z$ and $I = MR^2$, where $R = x_0$ is the distance from the axis of rotation. Thus

$$\boxed{L = M \, x_0^2 \, \omega_z \ .}$$

The angular momentum about the center of mass is computed using the vector equation $\vec{\mathbf{L}} = \vec{\mathbf{r}} \times \vec{\mathbf{p}}$, so

$$\boxed{\vec{\mathbf{L}} = [x_0, 0, z_0] \times M[0, \omega_z\, x_0, 0] = M[-\omega_z\, x_0\, z_0\,, 0\,, \omega_z\, x_0^2\, 0]\ .}$$

(c) Since the meteoroid is in empty space with no forces acting on it, there will be no torque acting on it, and its angular momentum vector $\vec{\mathbf{L}}$ will remain constant. The angular velocity ω does not then also remain constant, as can be seen by a "proof by contradiction." That is, suppose that ω were constant. Then the meteriod would be rotating about a fixed axis. Since $\vec{\mathbf{L}}$ is determined by $\vec{\boldsymbol{\omega}}$ and the geometry and mass distribution of the object, the rotation of the object about a fixed axis would imply that $\vec{\mathbf{L}}$ would rotate about the fixed axis, too. (See for example Problem 9D.1.) But if $\vec{\mathbf{L}}$ rotates about an axis which is not parallel to itself, then it must change, but we know from conservation of angular momentum that $\vec{\mathbf{L}}$ does not change. Thus, $\vec{\boldsymbol{\omega}}$ cannot be constant. The precise description of the motion gets rather complicated, but it can generally be described as "tumbling."

Gyroscopes:

Problem 7 (SG) *STUDY:* Motion of a gyroscope

9D.4 (S) A gyroscope is a massive rapidly spinning wheel mounted on an axle of negligible mass. If such a device is held with the axle horizontal and supported at one end by a vertical post, what will happen when it is released? Assume the wheel has mass m and moment of inertia I, that it is spinning with angular velocity ω, and that the distance between the wheel and the supported end of the axle is ℓ.

Answer: See answer in Study Guide.

Problem 8 (Y&F): Precession of a gyroscope and the force on its pivot

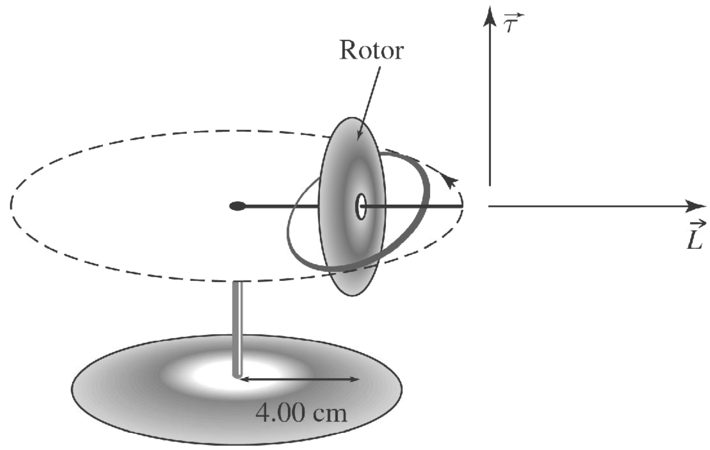
10.48: a) Since the gyroscope is precessing in a horizontal plane, there can be no net vertical force on the gyroscope, so the force that the pivot exerts must be equal in magnitude to the weight of the gyroscope,

$F = \omega = mg = (0.165\ \text{kg})(9.80\ \text{m/s}^2) = 1.617\ \text{N},\ 1.62\ \text{N}$ to three figures.

b) Solving Eq. (10.36) for ω,

$$\omega = \frac{\omega R}{I\Omega} = \frac{(1.617\ \text{N})(4.00\times10^{-2}\ \text{m})}{(1.20\times10^{-4}\ \text{kg}\cdot\text{m}^2)(2\pi)\text{rad}/2.20\ \text{s}} = 188.7\ \text{rad/s},$$

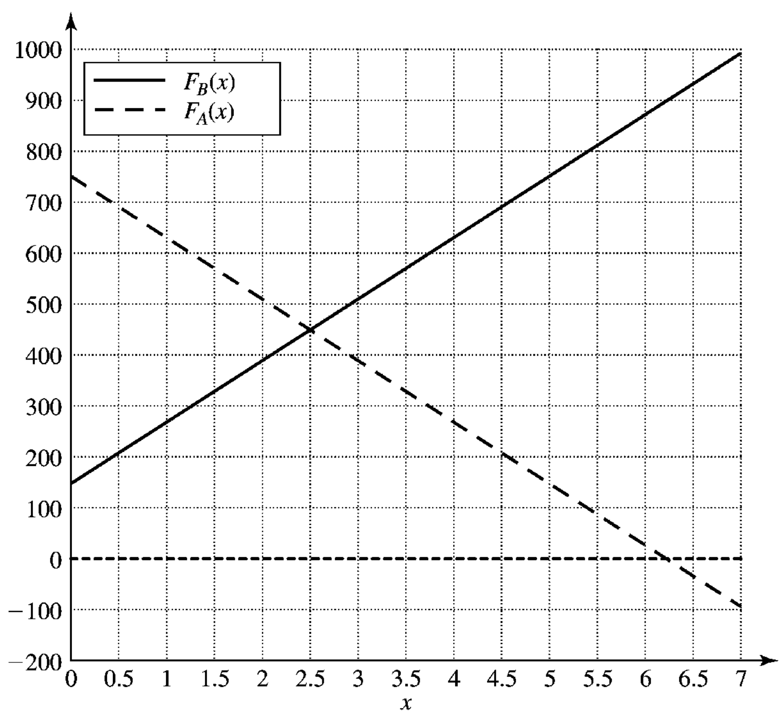
which is 1.80×10^3 rev/min. Note that in this and similar situations, since Ω appears in the denominator of the expression for ω, the conversion from rev/s and back to rev/min *must* be made.



Rotational Equilibrium:

Problem 9 (Y&F): When will the beam tip?

11.12: a)



b) $x = 6.25$ m when $F_A = 0$, which is 1.25 m beyond point B. c) Take torques about the right end. When the beam is just balanced, $F_A = 0$, so $F_B = 900$ N. The distance that point B must be from the right end is then $\frac{(300\text{ N})(4.50\text{ m})}{(900\text{ N})} = 1.50$ m.

Problem 10 (Y&F): Balancing pieces of steel on the edge of a table

11.46: a) Denote the weight per unit length as α, so $w_1 = \alpha(10.0\,\text{cm})$, $w_2 = \alpha(8.0\,\text{cm})$, and $w_3 = \alpha l$.

The center of gravity is a distance x_{cm} to the right of point O where

$$x_{\text{cm}} = \frac{w_1(5.0\,\text{cm}) + w_2(9.5\,\text{cm}) + w_3(10.0\,\text{cm} - l/2)}{w_1 + w_2 + w_3}$$

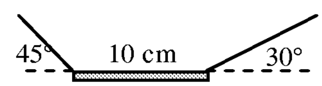
$$= \frac{(10.0\,\text{cm})(5.0\,\text{cm}) + (8.0\,\text{cm})(9.5\,\text{cm}) + l(10.0\,\text{cm} - l/2)}{(10.0\,\text{cm}) + (8.0\,\text{cm}) + l}.$$

Setting $x_{\text{cm}} = 0$ gives a quadratic in l, which has as its positive root $l = 28.8\,\text{cm}$.

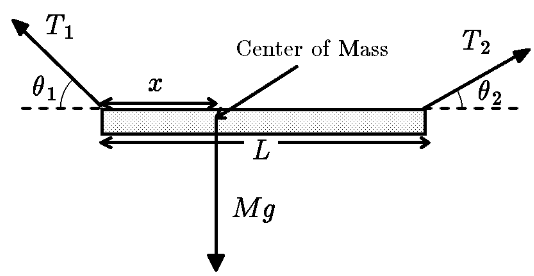
b) Changing the material from steel to copper would have no effect on the length l since the weight of each piece would change by the same amount.

Problem 11 (SG): Torques on a non-uniform rod

9A.6 (H) A non-uniform horizontal rod is supported by two massless strings, one making an angle of 30° with the horizontal and one an angle of 45°. The rod is 10 cm long. Where is its center of mass?



Answer: The rod is supported by two strings at angles $\theta_1 = 45°$ and $\theta_2 = 30°$. The tensions of the two strings are T_1 and T_2, and the gravitational force, acting at the center of mass of the rod, is Mg downward, where M is the mass of the rod. Since the rod is stationary, all forces and torques must cancel. The total force in the vertical direction is



$$T_1 \sin\theta_1 + T_2 \sin\theta_2 - Mg = 0\ , \tag{1}$$

and the total force in the horizontal direction is

$$T_1 \cos\theta_1 - T_2 \cos\theta_2 = 0\ . \tag{2}$$

Let the center of mass be a distance x from the left end of the rod (the point where the tension force of the first string acts). Then $L - x$ (with $L = 10$ cm being the length of the rod) is the distance of the other end of the rod (where the tension force of the second string acts) to the center of mass of the rod. The total torque about an axis through the center of mass is then given by

$$\tau = \sum_i R_i F_{i,\perp} = (L - x)T_2 \sin\theta_2 - xT_1 \sin\theta_1 = 0\ . \tag{3}$$

Note that gravity does not exert a torque around this axis because it acts at a point on the axis (the center of mass). We now have three equations for the three unknowns T_1, T_2, and x. Eq. (2) implies

$$T_2 = T_1 \frac{\cos\theta_1}{\cos\theta_2} \; .$$

Inserting this into Eq. (1) yields

$$T_1 \left(\sin\theta_1 + \frac{\cos\theta_1}{\cos\theta_2} \sin\theta_2 \right) = Mg \qquad \Longrightarrow \qquad T_1 = \frac{Mg\cos\theta_2}{\sin(\theta_1 + \theta_2)} \; ,$$

and plugging this back into Eq. (2) implies

$$T_2 = \frac{Mg\cos\theta_1}{\sin(\theta_1 + \theta_2)} \; .$$

Eq. (3) then yields

$$x = \frac{LT_2\sin\theta_2}{T_1\sin\theta_1 + T_2\sin\theta_2} = \frac{LT_2\sin\theta_2}{Mg} = L\frac{\cos\theta_1\sin\theta_2}{\sin(\theta_1+\theta_2)} = 3.7 \text{ cm} \; ,$$

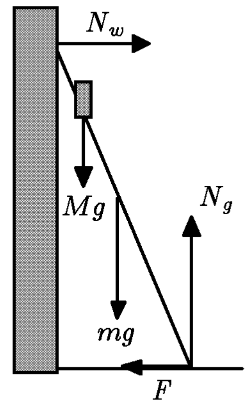
i.e. the center of mass is 3.7 cm away from the left end of the rod.

Problem 12 (SG): Will the ladder slip?

9A.7 (H) A uniform ladder of mass m and length ℓ rests against a smooth wall. A do-it-yourself enthusiast of mass M stands on the ladder a distance d from the bottom. If the ladder makes an angle θ with the ground, what is the minimum coefficient of friction required between the ladder and the ground in order that the ladder will not slip? Assume that there is negligible friction between the ladder and the wall.

Suppose that the homeowner in question is attempting to do some emergency repairs after her house has been damaged by a storm. Her 3 m ladder is not really long enough for the task, and she has to stand on the top rung, 2.8 m from the base, in order to do the job. Her mass is 70 kg, the mass of the aluminum ladder is negligible, and the angle the ladder makes with the horizontal is 70°. Unfortunately, due to the rain, the coefficient of friction between the ladder and her paved yard is only 0.20. Will she slip? Will the situation be improved if her friend, who has a mass of 90 kg but is afraid of heights, stands on the bottom rung of the ladder, 0.2 meters from the base, while she climbs to the top?

Answer: As long as the ladder does not slip it is stationary and all forces and torques acting on it must cancel. The forces acting on the ladder are a horizontal normal force N_w due to the wall acting at the top of the ladder, a vertical normal force N_g due to the ground acting at the bottom of the ladder, a horizontal static friction force $F \leq \mu_s N_g$, the vertical gravitational force mg due to the mass m of the ladder acting at the center of mass of the ladder, and finally a vertical normal force Mg due to the person of mass M acting at the point where the person stands. Since we are interested in the minimal coefficient of static friction μ_s, we can assume that $F = \mu_s N_g$. Equilibrium of forces in the horizontal direction implies



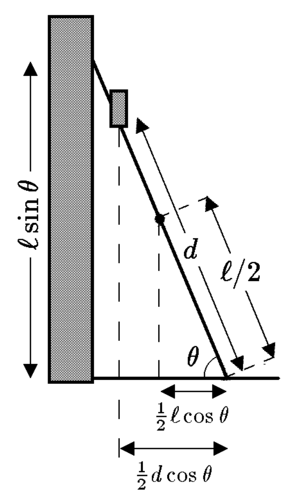
$$N_w - F = 0 \ ,$$

and equilibrium of forces in the vertical direction takes the form

$$N_g - mg - Mg = 0 \ .$$

We also need to cancel all torques about some arbitrarily chosen axis. It is convenient to choose an axis through the bottom end of the ladder, because both the normal force N_g and the friction F do not cause a torque around that axis (they act at a point on the axis). The remaining torques are

$$-N_w \ell \sin\theta + mg\frac{\ell}{2}\cos\theta + Mgd\cos\theta = 0 \ .$$



Note that the gravitational force mg acts at the center of mass a distance $\ell/2$ away from the axis, while the normal force Mg acts at the point where the person stands, which is a distance d from the axis. The normal force N_w tries to rotate the ladder clockwise and hence causes a negative torque, while the gravitational force and the normal force due to the person try to rotate the ladder counter-clockwise and thus cause a positive torque. We now have three equations for the three unknowns N_w, N_g, and F, which are straightforward to solve. One finds

$$N_g = (m+M)g, \qquad F = N_w = \frac{\left(\frac{1}{2}m\ell + Md\right) g\cos\theta}{\ell\sin\theta} \ ,$$

which implies

$$\mu_s = \frac{F}{N_g} = \frac{\left(\frac{1}{2}m\ell + Md\right)\cos\theta}{(m+M)\ell\sin\theta} \ .$$

Inserting the numerical values $\ell = 3$ m, $d = 2.8$ m, $M = 70$ kg, $m = 0$, and $\theta = 70°$, one obtains $\mu_s = 0.34$, which exceeds the assumed coefficient of friction of 0.2. Hence, the ladder will slip.

To take into account the friend of mass $M' = 90$ kg at a distance $d' = 0.2$ m from the base, we must modify two of the above equations to

$$N_g - mg - Mg - M'g = 0 \ ,$$

and

$$-N_w \ell \sin\theta + mg\frac{\ell}{2}\cos\theta + Mgd\cos\theta + M'gd'\cos\theta = 0 \ .$$

This implies that now

$$\mu_s = \frac{\left(\frac{1}{2}m\ell + Md + M'd'\right)\cos\theta}{(m + M + M')\ell\sin\theta} \ .$$

Again inserting the numerical values, we now find $\mu_s = 0.16$ which is less than 0.2. Hence, indeed the situation does improve.