MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

PROBLEM SET 11 SOLUTIONS

April 28, 2005

Applications of Kepler's Laws:

Problem 1 (Y&F): Planet Vulcan

12.32: Apply Newton's second law to Vulcan.

$$\sum F = ma : \frac{Gm_s m_v}{r^2} = m_v \frac{v^2}{r}$$

$$v = \frac{2\pi r}{T}$$

$$\frac{Gm_s}{r} = \left(\frac{2\pi r}{T}\right)^2$$

$$T = \sqrt{\frac{4\pi^2 r^3}{Gm_s}}$$

$$= \sqrt{\frac{4\pi^2 \left[\frac{2}{3}(5.79\times10^{10}\,\mathrm{m})\right]^3}{(6.67\times10^{-11}\,\mathrm{Nm^2/kg^2})(1.99\times10^{30}\,\mathrm{kg})}}$$

$$= 4.14\times10^6\,\mathrm{s}\left(\frac{1\mathrm{d}}{86{,}400\,\mathrm{s}}\right) = 47.9\text{ days}$$

Problem 2 (Y&F): Determining the mass of the Sun from the orbit of Venus

12.34: From either Eq. (12.14) or Eq. (12.19),

$$m_S = \frac{4\pi^2 r^3}{GT^2} = \frac{4\pi^2(1.08\times10^{11}\,\mathrm{m})^3}{(6.673\times10^{-11}\,\mathrm{N\cdot m^2/kg^2})\,((224.7\,\mathrm{d})(8.64\times10^4\,\mathrm{s/d}))^2}$$

$$= 1.98\times10^{30}\,\mathrm{kg}.$$

Problem 3 (Y&F): The orbit of *Helios B* around the Sun

12.37: a) For a circular orbit, Eq. (12.12) predicts a speed of

$$\sqrt{(6.673\times10^{-11}\,\mathrm{N\cdot m^2/kg^2})(1.99\times10^{30}\,\mathrm{kg})/(43\times10^9\,\mathrm{m})} = 56\,\mathrm{km/s}.$$

The speed doesn't have this value, so the orbit is not circular. b) The escape speed for any object at this radius is $\sqrt{2}(56\,\mathrm{km/s}) = 79\,\mathrm{km/s}$, so the spacecraft must be in a closed elliptical orbit.

More Sophisticated Orbit Problems:

Problem 4 (SG) *STUDY:* Kepler's second law of planetary motion and the conservation of angular momentum

9B.2 (S) One of the first achievements of the modern scientific method was Johannes Kepler's analysis of Tycho Brahe's observations of planetary positions. Kepler deduced three empirical laws governing planetary motion, from which Newton subsequently developed his law of universal gravitation.

Kepler's second law states that "a line drawn from the Sun to any planet sweeps out equal areas in equal times". Show that this law implies that the angular momentum of the planet due to its orbital motion is constant.

Answer: See the complete solution in the *Study Guide.*

Problem 5 (SG) *STUDY:* Comet orbits

9B.3 (S) Comets, in contrast to planets, follow very elongated elliptical orbits, so that their distance from the Sun varies greatly over the course of the orbit. Suppose that a short-period comet has an orbit whose closest approach to the Sun (or perihelion) is 5×10^{10} m, the furthest point (or aphelion) being 2×10^{12} m. What is its speed at perihelion?

Answer: See the complete solution in the *Study Guide.*

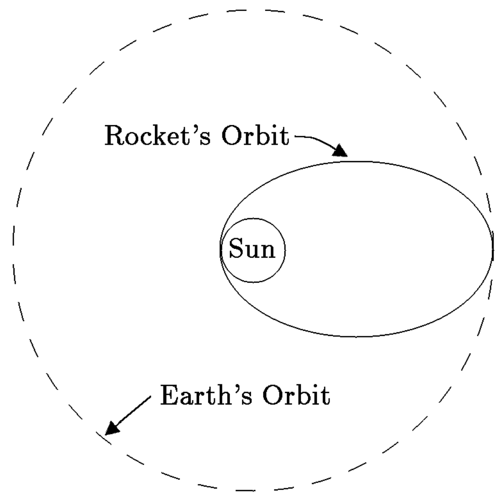
Problem 6 (SG): Disposing of nuclear waste in the Sun

9B.4 (H) Suppose that we wished to dispose of nuclear waste by launching it into space and 'firing it into the Sun'. One simple plan would be to start by putting the rocket into a circular orbit about the Sun at the same radius (and hence the same orbital speed) as the Earth. The rocket would then fire its engines in reverse for a short burst, slowing the rocket just enough so that the resulting orbit would graze the surface of the Sun, causing the rocket to be vaporized. By how much would the speed have to be changed? For comparison, if the rocket fired its engines forward for a short burst, by how much would the speed have to be changed for the rocket to leave the solar system altogether? [The Earth's orbit has a radius of 150 million kilometers and the Sun's radius is about 700 thousand kilometers. The mass of the Sun is 2.0×10^{30} kg, and the value of the gravitational constant G is $6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$.]

Challenge Problem:

The method of reaching the Sun described above is far from being the most fuel-efficient. Find a flight plan by which the rocket could reach the Sun with a significantly smaller change in speed required from the engines, and thus with significantly less fuel consumption.

Answer: This problem can be solved using energy and angular momentum conservation. Let us assume that the rocket first orbits the Sun together with the Earth at a radius $R = 150 \times 10^6$ km. It takes one year to complete an orbit around the Sun, i.e. the angular velocity is $\omega = 2\pi/\text{year}$. The corresponding linear velocity is $v = R\omega$. Now the rocket fires its engine in reverse and thereby decreases its linear velocity to v'. After that it has an angular momentum L about the Sun given by



$$L = Rmv' \ ,$$

where m is the mass of the rocket. The total energy of the rocket is the sum of its kinetic and potential energies

$$E = \frac{1}{2}mv'^2 - \frac{GMm}{R} \ ,$$

where $M = 2 \times 10^{30}$ kg is the mass of the Sun and $G = 6.67 \times 10^{-11}\ \mathrm{N \cdot m^2/kg^2}$ is the gravitational constant. After firing its engines the rocket is now in an elliptical orbit around the Sun. We want it to touch the Sun's surface at the point of the ellipse closest to the Sun. At that point the distance to the center of the Sun is given by the Sun's radius $r = 700,000$ km. At that point the angular momentum of the rocket takes the form

$$L = rmv'' \ ,$$

where v'' is the velocity of the rocket when it touches the Sun's surface. The energy of the rocket then takes the form

$$E = \frac{1}{2}mv''^2 - \frac{GMm}{r} \ .$$

Angular momentum conservation thus implies

$$v'' = \frac{Rv'}{r} \ ,$$

and energy conservation implies

$$\frac{1}{2}v'^2 - \frac{GM}{R} = \frac{1}{2}v''^2 - \frac{GM}{r} = \frac{R^2}{2r^2}v'^2 - \frac{GM}{r} \ ,$$

which leads to

$$v' = \sqrt{\frac{2GMr}{R(R+r)}} \ .$$

Inserting the appropriate numerical values, the change in speed of the rocket is given by

$$w = v' - v = -27 \text{ km/s} .$$

To leave the solar system, the total energy of the rocket needs to be changed to zero. The required speed v''' is thus given by

$$\frac{1}{2}mv'''^2 - \frac{GMm}{R} = 0 \quad \Longrightarrow \quad v''' = \sqrt{\frac{2GM}{R}} .$$

The change in speed required to leave the solar system is thus given by

$$w' = v''' - v = 12 \text{ km/s} .$$

The above strategy to dispose of nuclear waste is much harder to realize than to leave the solar system entirely (which is another way to get rid of the waste). However, it is possible for the rocket to reach the Sun with less fuel consumption than used with the method described above. The rocket can fire its engines forward and almost — but not quite — catapult itself out of the solar system. At the farthest point from the Sun — thanks to the large distance from the Sun — the rocket can change its angular momentum by a large amount by changing its velocity just a little bit. A small burst of the rocket can bring the angular momentum to zero, and then the rocket will fall straight into the Sun.

Black Holes:

Problem 7 (Y&F): Black hole at the center of the Milky Way galaxy

12.43: a) From Eq. (12.12),

$$M = \frac{Rv^2}{G} = \frac{(7.5\text{ ly})(9.461\times10^{15}\text{ m/ly})(200\times10^3\text{ m/s})^2}{(6.673\times10^{-11}\text{ N}\cdot\text{m}^2/\text{kg}^2)}$$
$$= 4.3\times10^{37}\text{ kg} = 2.1\times10^7\ M_S.$$

b) It would seem not.

c) $$R_S = \frac{2GM}{c^2} = \frac{2v^2R}{c^2} = 6.32\times10^{10}\text{ m},$$

which does fit.

Problem 8 (Y&F): Tidal forces near a black hole

12.87: a) There are many ways of approaching this problem; two will be given here.

I) Denote the orbit radius as r and the distance from this radius to either ear as δ. Each ear, of mass m, can be modeled as subject to two forces, the gravitational force from the black hole and the tension force (actually the force from the body tissues), denoted by F. Then, the force equations for the two ears are

$$\frac{GMm}{(r-\delta)^2} - F = m\omega^2(r-\delta)$$
$$\frac{GMm}{(r+\delta)^2} + F = m\omega^2(r+\delta),$$

where ω is the common angular frequency. The first equation reflects the fact that one ear is closer to the black hole, is subject to a larger gravitational force, has a smaller acceleration, and needs the force F to keep it in the circle of radius $r-\delta$. The second equation reflects the fact that the outer ear is further from the black hole and is moving in a circle of larger radius and needs the force F to keep in in the circle of radius $r+\delta$.

Dividing the first equation by $r-\delta$ and the second by $r+\delta$ and equating the resulting expressions eliminates ω, and after a good deal of algebra,

$$F = (3GMm\delta)\frac{(r+\delta)}{(r^2-\delta^2)^2}.$$

At this point it is prudent to neglect δ in the sum and difference, but recognize that F is proportional to δ, and numerically $F = \frac{3GMm\delta}{r^3} = 2.1$ kN. (Using the result of Exercise 12.39 to express the gravitational force in terms of the Schwartzschild radius gives the same result to two figures.)

II) Using the same notation,

$$\frac{GMm}{(r+\delta)^2} - F = m\omega^2(r+\delta),$$

where δ can be of either sign. Replace the product $m\omega^2$ with the value for $\delta = 0$, $m\omega^2 = GMm/r^3$ and solve for

$$F = (GMm)\left[\frac{r+\delta}{r^3} - \frac{1}{(r+\delta)^2}\right] = \frac{GMm}{r^3}\left[r+\delta - r(1+(\delta/r))^{-2}\right].$$

Using the binomial theorem to expand the term in square brackets in powers of δ/r,

$$F = \frac{GMm}{r^3}\left[r+\delta - r(1-2(\delta/r))\right] = \frac{GMm}{r^3}(3\delta),$$

the same result as above.

Method (I) avoids using the binomial theorem or Taylor series expansions; the approximations are made only when numerical values are inserted and higher powers of δ are found to be numerically insignificant.
Method (II) uses the fact even though the center of gravity is not at the center of mass, their separation is small compared to r, and so the period of the orbit of a point mass at that position can be used to characterize the motion.

If noninertial frames are allowed, the same result may be obtained by considering the frame of the astronaut; the difference in the position of the ears contributes a difference between the gravitational forces of magnitude $\frac{2GMm\delta}{r^3}$ and the fact that the astronaut is in a frame that rotates with frequency ω as found above means that in her frame the ears need an extra force of $m\omega^2\delta$, and the tension is the same. (This method was suggested by a first-year MIT undergraduate.)

This tension is much larger than that which could be sustained by human tissue, and the astronaut is in trouble. b) See the discussion above; the center of gravity is not the center of mass.

Simple Harmonic Motion:

Problem 9 (Y&F): The simple harmonic motion of an oscillating toy

13.27: a) $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = 0.0284\ \text{J}$.

b) $\sqrt{x_0^2 + \frac{v_0^2}{\omega^2}} = \sqrt{(0.012\ \text{m})^2 + \frac{(0.300\ \text{m/s})^2}{(300\ \text{N/m})/(0.150\ \text{kg})}} = 0.014\ \text{m}.$

c) $\omega A = \sqrt{k/m}A = 0.615\ \text{m/s}\cdot$

Problem 10 (SG) *STUDY:* Pendulum motion

2D.2 (S) (a) A pendulum consists of a 250 g bob suspended by a string of negligible mass. If I hold the bob so that the string is taut and makes an angle θ with the vertical, and then release it, what forces are acting at the moment of release?

(b) If I agree to displace the bob by no more than 5° from the vertical, find a differential equation that describes (to a good approximation) the subsequent motion of the bob. Show that this equation is satisfied if the motion of the bob takes the form

$$\theta = A\sin(\omega t + \phi)$$

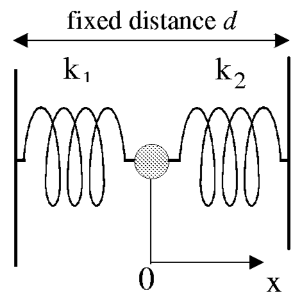
where A and ϕ are arbitrary constants, and derive an expression for ω. Describe in words the motion of the bob.

(c) I wish to use the pendulum to drive a clock. What length of string should I use to get a period of 1.00 s, and how fast will the bob then move at the bottom of its sweep if the maximum angle is $\pm 2°$?

Answer: See the complete solution in the *Study Guide.*

Problem 11 (SG): A mass between two springs

2D.3 (H) A small object of mass m is held by two springs with spring constant k_1 and k_2 as shown. The mass rests on a smooth surface so that the effects of friction are negligible. Left undisturbed, the mass sits at position $x = 0$.



(a) If it is now displaced to position $x = -A$ and released, what forces are acting on it at the instant of release?

(b) Obtain a differential equation describing the motion of the mass after its release, and use this to derive an expression for its period of oscillation.

(c) If I turn the device on end, so that the springs hang vertical, what is the effect on the position of the stable configuration and on the motion of the mass when displaced? Assume the springs themselves have negligible mass.

a) When m is at position $x = 0$, let the extension (from the unstretched length) of spring 1 be e_1 and of spring 2 be e_2.

there is no net force on m when it is at $x = 0$

$$\therefore \quad -k_1 e_1 + k_2 e_2 = 0 \quad \ldots\ldots (1)$$

when m is at position x the net force on the mass in the $+\hat{x}$-direction is

$$F_x = -k_1(e_1 + x) + k_2(e_2 - x) \quad \ldots\ldots (2)$$

or $$F_x = -k_1 x - k_2 x \quad \ldots (3)$$ (we have used result (1))

For $x = -A$ we obtain

$$F_x = (k_1 + k_2)A$$

i.e net force is $\underline{\underline{(k_1+k_2)A}}$ in the $+\hat{x}$-direction

b) For any arbitrary displacement x of the mass (from equilibrium) the force is $F_x = -(k_1+k_2)x$ see equ. (3)

Therefore the acceleration $\frac{d^2x}{dt^2}$ of the mass at that instant is

$$\frac{d^2x}{dt^2} = -\frac{(k_1+k_2)}{m}x \quad \ldots (4)$$

This is the equation of motion of m.

The solution of (4), with boundary conditions $x = -A$ and $\frac{dx}{dt} = 0$ at $t=0$, is

$$x = -A \cos\sqrt{\frac{k_1+k_2}{m}}\,t \quad \ldots (5)$$

Note: You can verify that this is the correct solution by differentiating (5) twice. You will find that it does satisfy equation (4).

Also for $t = 0$ (5) gives $x=-A$, $\frac{dx}{dt}=0$.

The period of $\cos\omega t$ is $T = \frac{2\pi}{\omega}$

(note: $\cos\left[\omega\left(t+\frac{2\pi}{\omega}\right)\right] = \cos[\omega t + 2\pi]$
$= \cos\omega t$)

b) contd.

∴ period of oscillation of the mass is $T = 2\pi\sqrt{\dfrac{m}{k_1+k_2}}$

c) If the device is on end there is one more force acting on the mass — the force of gravity mg acting downwards. We will define a new coordinate system with vertically up defined as the $\hat{y}$-direction, and $y=0$ the position of the mass when the net force due to the springs is zero.

By analogy with equation (3) when the mass is at position y the net force F_y on the mass in the $\hat{y}$ direction is

$$F_y = -(k_1+k_2)y - mg$$

c) contd.

At equilibrium $F_y = 0$

$$\therefore \quad -(k_1+k_2)\,y_{equilibrium} - mg = 0$$

$$\text{or} \quad y_{equilibrium} = -\frac{mg}{k_1+k_2}$$

ie the mass will be in equilibrium a distance $\underline{\underline{\dfrac{mg}{k_1+k_2}}}$ below the original position of equilibrium.

If we now displace m from this new position of equilibrium by a distance y' in the $+\hat{j}$ direction the net force on the mass in the $+\hat{j}$-direction will be

c) contd.

$$F_{y'} = -(k_1+k_2)(y_{equilibrium} + y') - mg = -(k_1+k_2)y'$$

$$\therefore \frac{d^2y'}{dt^2} = -\frac{(k_1+k_2)}{m} y'$$

this is the same equation of motion as we found in b). ie the resultant motion will have the same period.

Problem 12 (SG) ***STUDY:*** **Mass sliding in a spherical bowl**

4E.2 (S) (a) The interior of an ornamental bowl forms a hemisphere of radius 10 cm. The surface is smooth and highly polished. If I place a small object at a distance s from the bottom of the bowl (where s is measured along the inner surface of the bowl, so that s varies from 0 to $\pi R/2$, $R = 10$ cm), how fast will it be moving when it reaches the bottom of the bowl?

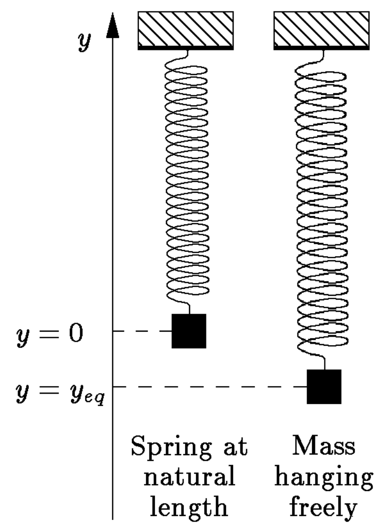
(b) What is the form of the potential energy of the object if s is small? Deduce the force acting on the object if it is displaced slightly from the bottom of the bowl and then released, and describe the resulting motion.

Answer: See the complete solution in the *Study Guide.*

More Complicated Oscillators:

Problem 14 (SG): A mass suspended on a spring

4E.5 (H) A 1 kg mass is suspended from a spring of spring constant 120 N/m. The coordinate system is defined so that y is directed vertically upwards and $y = 0$ when the spring is at its natural length. The mass is first positioned at $y = 0$, and then lowered gently until it is hanging freely from the spring.

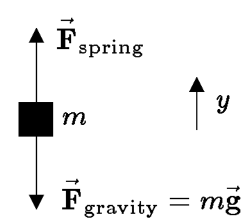


Where does the mass come to rest? What then is its gravitational potential energy, and what is the potential energy stored in the spring, compared to their values when the mass is at $y = 0$? Explain why these two quantities are not equal. What work was done (i) by gravity, (ii) by the spring force, as the mass was lowered from $y = 0$ to its equilibrium position? If instead of being lowered gently the mass had simply been released at $y = 0$ and allowed to fall under the combined effects of gravity and the spring force, at what speed would it be moving when it crosses the equilibrium value of y, and what would be its subsequent motion (be quantitative as far as possible)?

Answer:

When the mass comes to rest there is no net force acting on it, so

$$\vec{\mathbf{F}}_{\text{spring}} + m\vec{\mathbf{g}} = 0 \ .$$



By Hooke's law, the y-component of this equation becomes:

$$F_y = -ky_{\text{eq}} - mg = 0 \ ,$$

so

$$y_{eq} = \boxed{-\frac{mg}{k}}$$

$$= -\frac{1\ \text{kg} \times 9.8\ \text{m/s}^2}{120\ \text{N/m}} = -\frac{1\ \text{kg} \times 9.8\ \text{m/s}^2}{120\ (\text{kg} \cdot \text{m/s}^2)/\text{m}} = -0.082\ \text{m} = \boxed{-8.2\ \text{cm.}}$$

The gravitational potential energy of the mass is given by

$$U_{\text{grav}}(y_{\text{eq}}) = U_{\text{grav}}(0) - \int_0^{y_{\text{eq}}} \vec{\mathbf{F}}_{\text{grav}} \cdot d\vec{\mathbf{r}} = U_{\text{grav}}(0) - \int_0^{y_{\text{eq}}} (-mg)dy = U_{\text{grav}}(0) + mgy_{\text{eq}} \ .$$

Therefore

$$U_{\rm grav}(y_{\rm eq}) - U_{\rm grav}(0) = mgy_{\rm eq} = 1\ {\rm kg} \times 9.8\ {\rm m/s^2} \times (-0.082\ {\rm m}) = \boxed{-0.80\ {\rm J.}}$$

For the potential energy stored in the spring,

$$U_{\rm spring}(y_{\rm eq}) = U_{\rm spring}(0) - \int_0^{y_{\rm eq}} \vec{\mathbf{F}}_{\rm spring} \cdot {\rm d}\vec{\mathbf{r}} = U_{\rm spring}(0) - \int_0^{y_{\rm eq}} (-ky)\,{\rm d}y\ .$$

Therefore

$$U_{\rm spring}(y_{\rm eq}) - U_{\rm spring}(0) = \frac{1}{2}ky_{\rm eq}^2 = \frac{1}{2} \times 120\ {\rm N/m} \times (-0.082\ {\rm m})^2 = \boxed{0.40\ {\rm J}\ .}$$

The magnitudes of the two potential energies are not equal, because the spring force decreases as $|y|$ decreases, so in the region $0 \le |y| \le |y_{\rm eq}|$ the spring force is smaller in magnitude than the gravitational force. As the mass is lowered, the magnitude of the work done by the spring force is therefore less than that of gravity, so the magnitude of the potential energy stored in the spring is less than that of gravity. However, since the kinetic energy of the mass is zero both before and after it is lowered, the work-energy theorem tells us that the total work done on the mass during the lowering process must be zero. The extra contribution to the total work is provided by the person lowering the mass "gently" to its equilibrium point. This person exerts an additional upward force to prevent the mass from falling freely, so she does negative work as the mass is lowered.

(i) The work done by gravity is equal to *minus* the change of gravitational potential energy. So gravity does 0.80 J of work on the mass as it is lowered.

(ii) The work done by the spring as the mass is lowered is minus the change in the spring's potential energy, or -0.40 J.

If the mass were allowed to fall from rest starting at $y = 0$, then its speed at $y_{\rm eq}$ can be calculated by conservation of energy. If we take $U_{\rm grav}(0) = U_{\rm spring}(0) = 0$, then the initial potential energy is zero. Since the initial kinetic energy is also zero, the initial total mechanical energy is zero. Thus, the total mechanical energy at $y = y_{\rm eq}$ must also be zero. So

$$\begin{aligned} 0 &= U_{\rm grav}(y_{\rm eq}) + U_{\rm spring}(y_{\rm eq}) + \frac{1}{2}mv^2 \\ &= -0.80\ {\rm J} + 0.40\ {\rm J} + \frac{1}{2}mv^2\ , \end{aligned}$$

and

$$v = \sqrt{0.80\ {\rm J/kg}} = \sqrt{0.80({\rm kg}\cdot{\rm m^2/s^2})/{\rm kg}} = \boxed{0.89\ {\rm m/s}\ .}$$

The subsequent motion can be found most clearly by defining a new variable, $y' = y - y_{\rm eq}$, so $y' = 0$ at the equilibrium position. Then if the mass is displaced from

$y' = 0$ there is a net restoring force $F_y(y') = -ky'$ in the y' (or y) direction. Thus the equation of motion for the mass m becomes

$$m\frac{d^2y'}{dt^2} = -ky' \; ,$$

or

$$\frac{d^2y'}{dt^2} = -\frac{k}{m}y' = -\omega^2 y' \; , \qquad \text{with } \omega = \sqrt{\frac{k}{m}} \; .$$

Therefore the subsequent motion of the mass will be simple harmonic motion with $\boxed{\text{amplitude 8.2 cm}}$ and

$$\text{period} = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{1\text{ kg}}{120\text{ N/m}}} = 2\pi\sqrt{\frac{1\text{ kg}}{120\ (\text{kg}\cdot\text{m/s}^2)/\text{m}}} = \boxed{0.57\text{ s}\; .}$$

Problem 15 (Y&F): A torsional pendulum

13.36: Solving Eq. (13.24) for κ in terms of the period,

$$\begin{aligned}\kappa &= \left(\frac{2\pi}{T}\right)^2 I \\ &= \left(\frac{2\pi}{1.00\text{ s}}\right)^2 ((1/2)(2.00\times10^{-3}\text{ kg})(2.20\times10^{-2}\text{ m})^2) \\ &= 1.91\times10^{-5}\text{ N}\cdot\text{m/rad}.\end{aligned}$$

Problem 16 (Y&F): The relation between a simple pendulum and a physical pendulum

13.47: For the situation described, $I = mL^2$ and $d = L$ in Eq. (13.39); canceling the factor of m and one factor of L in the square root gives Eq. (13.34).