MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

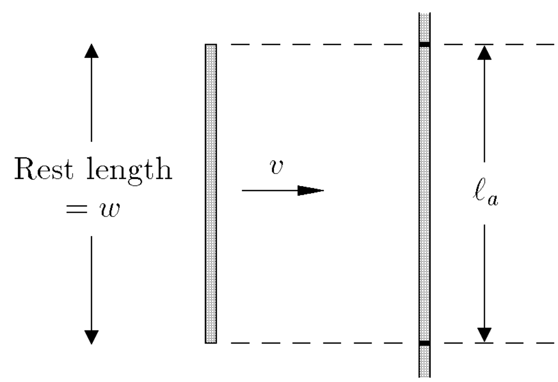
Physics 8.01 Spring 2005

PROBLEM SET 12 SOLUTIONS

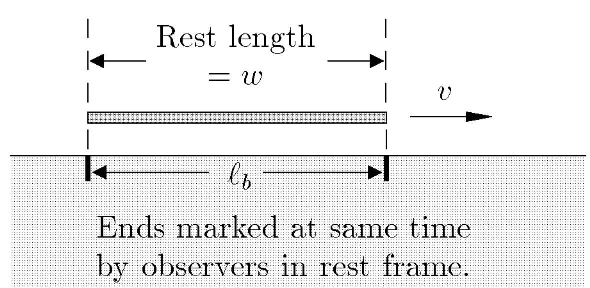
May 5, 2005

Problem 1: Moving rods and clocks

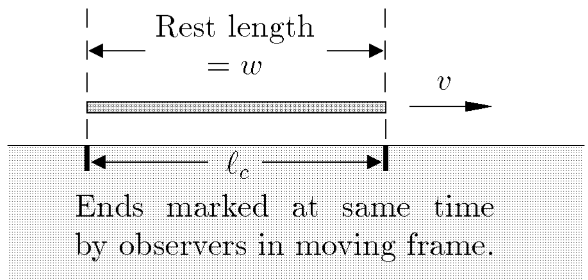
(a) A rod, with rest length w, moves with speed v in a direction perpendicular to its length. It passes a second, longer rod, which is at rest and is oriented parallel to the first rod. As it passes, observers who are at rest on the second rod mark the location of each end of the first rod. What is the distance l_a between these two marks, as measured in the stationary frame?



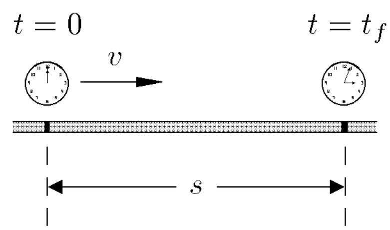
b) The same rod again moves with speed v, but this time it is moving parallel to its length, moving along the surface of a stationary table. At a certain prearranged time, observers in the rest frame of the table mark the instantaneous locations of each end of the rod. What is the distance ℓ_b between the two marks, as measured in the stationary frame?



c) The rod is still moving along the table, but this time there are travelers on the moving rod who have synchronized their clocks in the frame of the rod. At a prearranged time, travelers at each end of the rod reach out and make a mark on the table. What is the distance ℓ_c between these two marks, as measured in the stationary frame (i.e., the frame of the table)?



d) A clock moves along a track at speed v. The track is at rest, and has length s. The clock is set to zero at the start of the track. What time t_f does it read when it reaches the end of the track?



Answer:

(a) Since the rod is oriented perpendicular to the direction of motion, its length appears the same to observers in both frames. Therefore the marks on the table occur at a distance of

$$\boxed{\ell_a = w \quad .}$$

(b) Observations of distances parallel to the direction of motion are affected by the relative motion of the two frames. Observers on the table will see the rod contracted by the factor γ,

$$\boxed{\ell_b = \frac{w}{\gamma} \quad .}$$

(c) One way to see the answer to this problem is to move into the frame of the rod. Observers at the ends of the rod mark off a distance w on the table which moves past them with speed $-v$. This is equivalent to making a measurement of a hypothetical stick which sits on the table. Observers in the rod's frame see distances in the table frame contracted. Since the hypothetical stick appears to observers on the rod to be contracted to a distance w, we argue that the stick's length in the table frame must be

$$\boxed{\ell_c = \gamma w \quad .}$$

ALTERNATELY: This solution can be obtained quickly from the Lorentz transformation equations. Let primed quantities denote rod coordinates and unprimed quantities denote table coordinates. Then separation between the events, i.e. the two marks on the table, is obtained from

$$\Delta x = \gamma(\Delta x' + \beta c \Delta t') \quad .$$

$\Delta x' = \omega$ and $\Delta t' = 0$, giving

$$\ell_c = \gamma\omega \quad .$$

YET ANOTHER ALTERNATIVE: This problem can also be worked out in the frame of the table. The marks are made simultaneously in the frame of the rod, when clocks on both ends of the rod read the same time, call it t'. In the frame of the table, these clocks are do not appear synchronized; the trailing clock leads by $\beta w/c$. The table observers see the trailing clock mark the table at time t'. At this moment they observe the leading clock to be at a distance w/γ from the trailing clock and to read $t' - \beta w/c$. After the elapse of time $\beta w/c$ as measured on the leading clock, the leading clock will read t' and the second mark will be made. Since moving clocks run slowly, the time elapsed in the table frame between the making of the marks is $\gamma\beta w/c$. In this time, the rod has traveled a distance $v(\gamma\beta w/c) = \gamma\beta^2 w$. Putting this together, $\ell_c = w/\gamma + \gamma\beta^2 w = w/\gamma(1 + \gamma^2\beta^2) = \gamma w$.

(d) In the clocks frame the track has length s/γ. The clock sees the track moving under it with speed $-v$. The time it takes for the end of the track to reach the clock is then

$$\boxed{t_f = \frac{s}{\gamma v} \quad .}$$

Problem 2: The lifetime of a relativistic muon (Y&F:37.2)

The positive muon (μ^+), an unstable particle, lives on average 2.20×10^{-6} s (measured in its own frame of reference) before decaying.

(a) If such a particle is moving, with respect to the laboratory, with a speed of $0.900c$, what average lifetime is measured in the laboratory?

(b) What average distance, measured in the laboratory, does the particle move before decaying?

Answer:

37.2: a) $\gamma = \dfrac{1}{\sqrt{1-(0.9)^2}} = 2.29.$

$t = \gamma\tau = (2.29)\,(2.20\times10^{-6}\,\text{s}) = 5.05\times10^{-6}\ \text{s}.$

b) $d = vt = (0.900)\,(3.00\times10^{8}\ \text{m/s})\,(5.05\times10^{-6}\,\text{s}) = 1.36\times10^{3}\ \text{m} = 1.36\ \text{km}.$

Problem 3: Time as measured in relativistic spacecraft (based on Y&F:37.6)

While you are on a space station in deep space, far away from any gravitational fields, a race pilot flies past you in her space racer at a constant speed of $0.800c$ relative to you. At the instant the space racer passes you, both of you start timers at zero.

(a) There is a space buoy at rest relative to you, at a distance of 1.20×10^{8} m (as measured in your reference frame). If you looked at your timer at the same time (as measured in your coordinate system) that the space racer passes the buoy, what time would it read?

(b) When the space racer passes this buoy, what does the pilot read on her timer?

(c) When the race pilot passes the buoy, how far has she traveled since passing you, as measured in her reference frame?

(d) At the instant which, as judged by the space racer, is simultaneous to her passing the buoy, what is the reading on your timer?

Answer:

(a) In your coordinate system this is just a straightforward kinematics problem. The space racer is traveling a distance 1.20×10^{8} m at speed $0.800c$, so the time it takes is just

$$\Delta t = \frac{1.20\times10^{8}\ \text{m}}{0.800(3.00\times10^{8}\ \text{m/s})} = \boxed{0.500\ \text{s}\ .}$$

(b) Analyzing the situation in your reference frame, you would find that her timer would be running slowly by a factor of γ, where

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-(0.800)^2}} = 1.667\ .$$

So when your timer reads 0.500 s, hers will read

$$\Delta t' = \frac{0.500\ \text{s}}{1.667} = \boxed{0.300\ \text{s}\ .}$$

Alternatively, we can solve the problem in the space racer's frame of reference. She would measure the distance to the space buoy as contracted by a factor of γ, so the time it takes for her to reach it at $0.800c$ is

$$\Delta t' = \frac{(1.20\times10^{8}\ \text{m})/1.667}{0.800(3.00\times10^{8}\ \text{m/s})} = 0.300\ \text{s}\ ,$$

identical to the previous answer.

(c) In the space racer's reference frame, the distance to the buoy appears contracted by a factor of $\gamma = 1.667$, so it is

$$\Delta x' = \frac{1.20 \times 10^8 \text{ m}}{1.667} = \boxed{7.20 \times 10^7 \text{ m}} .$$

This number can also be calculated by multiplying the speed $0.800c$ times the time interval as measured on the space racer's timer, 0.300 s.

(d) Using only the reference frame of the space racer, recall that both timers were set to zero at the same event. The space racer's timer has since measured 0.300 s. In her reference frame your timer would appear to run slowly by a factor of $\gamma = 1.667$, so at the end of the time interval your timer would read

$$\Delta t'' = \frac{0.300 \text{ s}}{1.667} = \boxed{0.180 \text{ s}} .$$

Problem 4: An unstable particle produced in the upper atmosphere (Y&F:37.10, slightly reworded)

An unstable particle is created in the upper atmosphere from a cosmic ray and travels straight down toward the surface of the Earth with a speed of $0.99540c$ relative to the Earth. A scientist at rest on the Earth's surface measures that the particle is created at an altitude of 45.0 km.

(a) As measured by the scientist, how much time does it take the particle to travel the 45.0 km to the surface of the Earth?

(b) Use the length contraction formula to calculate the distance from where the particle is created to the surface of the Earth as measured in the particle's frame.

(c) In the particle's frame, how much time elapses from when the particle is created to when it strikes the surface of the Earth? Calculate this time both by the time dilation formula and also from the distance calculated in part (b). Do the two results agree?

Answer:

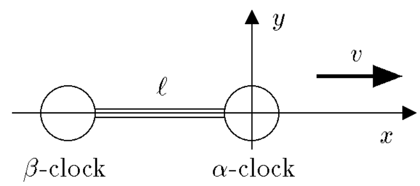
37.10: a) $t = \dfrac{4.50 \times 10^4 \text{ m}}{0.99540c} = 1.51 \times 10^{-4}$ s.

b) $\gamma = \dfrac{1}{\sqrt{1-(0.9954)^2}} = 10.44$

$$h' = \frac{h}{\gamma} = \frac{45 \text{ km}}{10.44} = 4.31 \text{ km}.$$

c) $\dfrac{h'}{0.99540c} = 1.44 \times 10^{-5}$ s, and $\dfrac{t}{\gamma} = 1.44 \times 10^{-5}$ s; so the results agree but the particle's lifetime is dilated in the frame of the earth.

Problem 5: A quantitative treatment of the relativity of simultaneity



Consider a device consisting of two clocks and a rod between them, as shown at the right. The rod is oriented along the x axis of a coordinate system, which we will refer to as the "laboratory frame." The whole device is moving to the right (positive x direction), relative to the laboratory frame, with speed v. We will call the leading clock the α-clock, and the rear clock the β-clock. The length of the rod in the rest frame of the device is ℓ_0, but in the laboratory frame is has length ℓ. The clocks in the laboratory frame are set so that $t = 0$ when the α-clock passes the origin.

(a) In terms of ℓ_0, v, and the speed of light c, what is the length ℓ, as measured in the laboratory frame?

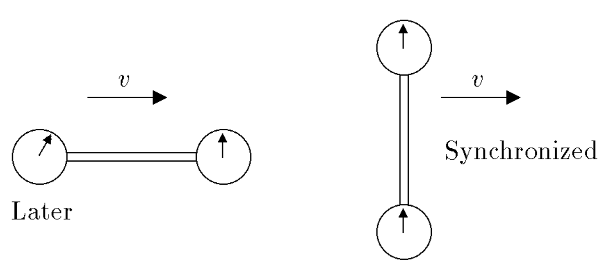
(b) Two observers ride on the moving device, one located at each clock, and they synchronize the two clocks using the definition of simultaneity appropriate to their frame of reference. There are many (equivalent) ways that they could do this, but we will suppose that at the instant when the α-clock passes the origin of our coordinate system, the observer at that clock sets the clock to zero, and emits a light pulse. When the light pulse reaches the β-clock, the observer on that clock sets it to ℓ_0/c, allowing for the light-travel time between the two clocks. In the laboratory frame, at what time t_f and location x_f does the light pulse reach the β-clock?

(c) In the laboratory frame of reference, what is the reading t_α on the α-clock at time t_f, the same instant when the light pulse reaches the β-clock?

(d) We let $t_\beta = \ell_0/c$ denote the time to which the β-clock is set when the light pulse reaches it, and we continue to use t_α to denote the reading on the α-clock at the same time, as measured in the laboratory frame. Then the quantity $t_\beta - t_\alpha$ indicates the error in synchronization of the two clocks, as seen in the laboratory frame. Show that this difference is given by

$$t_\beta - t_\alpha = \frac{v\ell_0}{c^2} .$$

That is, whenever a system that is moving relative to an observer contains clocks which are synchronized in the frame of the system and which are separated from each other in the direction of the motion, then in the observer's reference frame the trailing clock will read a later time than the leading clock by an offset $v\ell_0/c^2$. On the other hand, if the clocks are separated in a direction perpendicular to the motion, they will appear to be synchronized, as illustrated below:



The above statement, when combined with statements about time dilation and length contraction, give a full description of the kinematical consequences of special relativity.

Answer:

(a) ℓ is the length of a rod that is moving in the direction of its length, so it is Lorentz-contracted:

$$\boxed{\ell = \frac{\ell_0}{\gamma}\ ,}$$

where

$$\boxed{\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}\ .}$$

(b) In the laboratory frame, the light pulse leaves the origin at $t = 0$ and travels along the negative x-axis. So, since it travels at speed c, its position as a function of time is given by

$$x_{\text{pulse}} = -ct\ .$$

The β-clock is located at $t = 0$ at $x = -\ell = -\ell_0/\gamma$, and it travels along the positive x-axis with speed v. So its position as a function of time is given by

$$x_\beta = -\ell_0/\gamma + vt\ .$$

The light pulse will intesect the β-clock when $x_{\text{pulse}} = x_\beta$, which happens when

$$-ct = -\ell_0/\gamma + vt\ ,$$

or

$$\boxed{t_f = \frac{\ell_0}{\gamma(c + v)}\ .}$$

The x-coordinate of this intersection can be found easily from the x_{pulse} equation:

$$x_f = -ct_f = \boxed{-\frac{c\ell_0}{\gamma(c + v)}\ .}$$

(c) Described in the laboratory frame, the α-clock is set to zero at $t = 0$, and then runs slowly by a factor of γ. So at time t_f, it reads time

$$t_\alpha = \frac{t_f}{\gamma} = \frac{\ell_0}{\gamma^2(c + v)} = \frac{\ell_0(c^2 - v^2)}{c^2(c + v)} = \boxed{\frac{\ell_0(c - v)}{c^2}\ .}$$

(d) Since $t_\beta = \ell_0/c$, the difference is

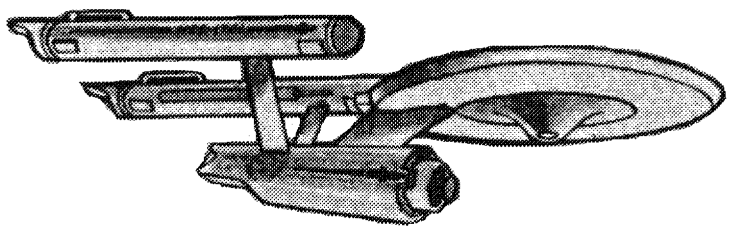
$$t_\beta - t_\alpha = \frac{\ell_0}{c} - \frac{\ell_0(c - v)}{c^2} = \boxed{\frac{v\ell_0}{c^2}\ ,}$$

as we were told to expect.

Problem 6: A high speed space chase

A badly programmed computer in the control system of the starship Excalibur goes berserk, launching the fully armed spacecraft from the planet Gamma Trianguli VI. The Excalibur moves at speed v_X along a straight line leading toward one of the most populated areas in the galaxy.

(a) When the Excalibur leaves Gamma Trianguli VI, the clocks on the planet read t_1. After traveling for a time t_2 as measured on its own clock, the computer transmits a message to Gamma Trianguli VI, threatening to destroy the planet Eminiar VII. If the message travels at the speed of light, what is the time t_3 on the clocks of Gamma Trianguli VI when the message is received?



(b) The starship Enterprise arrives at Gamma Trianguli VI at time t_E, measured on the planet's clocks. It immediately takes off after Excalibur at speed v_E, where fortunately $v_E > v_X$. It rapidly overtakes Excalibur. At the moment of interception, what is the time t_4, as measured in the coordinate system of Gamma Trianguli VI?

(c) As measured on the Excalibur clocks, how much time Δt_X elapses between its take-off from Gamma Trianguli VI and the moment of interception?

Answer:

(a) The clocks on the Excalibur measure a time interval t_2. Observers on Gamma Trianguli VI see the Excalibur's clocks run slowly by a factor

$$\gamma_X = \frac{1}{\sqrt{1-\beta_X^2}} \quad ,$$

where $\beta_X = v_X/c$, and thus measure a time interval $\gamma_X t_2$ since the Excalibur left the planet. In this time, the Excalibur traveled a distance of $v_X \gamma_X t_2$. A signal traveling at the speed of light from this distance will arrive at the planet when $t_3 - t_1 = \gamma_X t_2 + (v_X/c)\gamma_X t_2$, or $t_3 = t_1 + (1+\beta_X)\gamma_X t_2$. So,

$$\boxed{t_3 = t_1 + \sqrt{\frac{1+\beta_X}{1-\beta_X}}\, t_2 \quad .}$$

(b) We can write the trajectories of the two spaceships in the Gamma Trianguli VI frame and find the time at which they cross. Take the spatial origin of the coordinate system to be at Gamma Trianguli VI. Since the Excalibur takes off at time t_1 and moves at speed v_X, its trajectory is given by

$$x_X = v_X(t - t_1) \quad .$$

Similarly, the trajectory for the Enterprise can be written as

$$x_E = v_E(t - t_E) \quad .$$

We want to find the time t_4 when the trajectories cross, which can be found by solving

$$v_X(t_4 - t_1) = v_E(t_4 - t_E) \quad .$$

Solving this for t_4 gives,

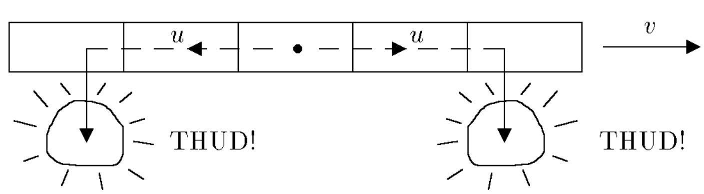
$$\boxed{t_4 = \frac{v_E t_E - v_X t_1}{v_E - v_X} \quad .}$$

(c) Recall that t_4, calculated above, is the ***time*** at which the interception occurs, as measured in the Gamma Trianguli VI coordinate system. Since the Excalibur took off at time t_1 in this system, the ***time interval*** is $t_4 - t_1$. According to observers in the Gamma Trianguli VI system, however, the clocks aboard the Excalibur are running slowly, by a factor of γ_X. Thus the clocks on Excalibur measure the shorter interval $\Delta t_X = (t_4 - t_1)/\gamma_X$. Using the expression for t_4 found in part (b) gives

$$\boxed{\Delta t_X = \frac{v_E(t_E - t_1)}{\gamma_X(v_E - v_X)} \quad .}$$

Problem 7: Dynamic Duo jumps off a fast train

Batman and Robin are standing next to each other on the roof of a train, which is moving at a steady velocity of magnitude v. (Since this is a relativity problem, v is of course not negligible compared to c.) The superheroes simultaneously set their stop-watches to zero, and then by prearrangement they start running along the train in opposite directions at speed u, measured in the frame of reference of the train. They each run for a time Δt, as measured on their own stop-watches, and then they each jump from the train. They hit the ground with a sharp thud, and come instantly to a stop. Miraculously neither member of the Dynamic Duo is hurt.



(a) As measured on the ground, how far apart are the two points at which the pair hit the ground?

(b) As seen from the ground, do the two superheroes hit the ground simultaneously? If not, calculate the time difference.

Answer:

(a) This problem can be answered most easily by thinking in the frame of reference of the train. The stopwatches run slow in this frame by a factor of $1/\sqrt{1-u^2/c^2}$, so each superhero runs a distance

$$\Delta x = \frac{u\Delta t}{\sqrt{1-u^2/c^2}} \ .$$

The distance between the two points at which the pair jump is then $2\Delta x$. Since both stopwatches run slowly by the same factor, in this frame of reference the two superheroes jump simultaneously. To calculate the distance between the two jumps as measured on the ground, one can imagine a long tape measure stretched along the ground. In the frame of the train this tape measure will appear compressed by a factor $1/\sqrt{1-v^2/c^2}$. Thus, the points at which the superheroes jump will be separated in their readings on the tape measure by

$$\boxed{\frac{2\Delta x}{\sqrt{1-v^2/c^2}} = \frac{2u\Delta t}{\sqrt{1-v^2/c^2}\sqrt{1-u^2/c^2}}} \ .$$

(b) We will continue to work in the frame of reference of the train. To calculate the time-separation of the jumps as seen from the ground, we can imagine two ground-based clocks, one at the site of each jump. These clocks are synchronized in the frame of reference of the ground, and the problem asks about the difference of the readings on these two clocks at the time of the jumps. Since the jumps are simultaneous in the frame of the train, the problem is simply one of calculating the apparent lack of synchronization of the ground-based clocks as seen in the train's frame of reference. According to special relativity, the "trailing" clock will read later. Since an observer on the train would see the ground moving from the front of the train to the back, it is the clock nearer the front of the train which reads later. The time difference is given by v/c^2 times the distance between the clocks, as measured in the frame of the clocks. This distance is just the answer to part (a). Thus an observer on the ground sees the superhero toward the front of the train jump ***later*** by an amount

$$\boxed{\frac{2uv\Delta t}{c^2\sqrt{1-u^2/c^2}\sqrt{1-v^2/c^2}}} \ .$$

ALTERNATIVE SOLUTION: This problem can also be worked in the frame of reference of the ground, so for comparison I will show how to do the calculation that way. I will let the ground-based coordinates be called (x,t), while the coordinates in the frame of reference of the train will be called (x',t'). The relation between the two frames is then given by the Lorentz transformation:

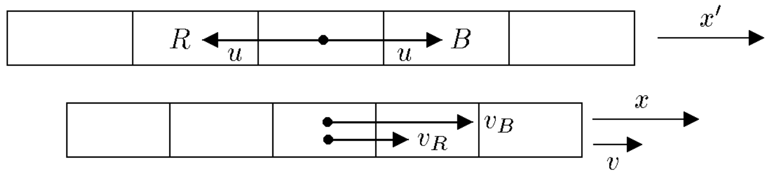
$$x = \gamma_v(x' + vt)$$

$$t = \gamma_v\left(t' + \frac{vx'}{c^2}\right),$$

where

$$\gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}} \ .$$

Suppose Batman runs toward the front of the train, while Robin runs the other way.



The key subtlety is that Batman's velocity in the unprimed frame is ***not*** $v+u$. (If velocities added so simply, then the speed of light would obviously not be invariant.) Instead we must use the formula for the relativistic addition of velocities, which implies that Batman's velocity in the ground frame is given by

$$v_B^x = \frac{u_B^x}{u_B^0} c = \frac{u+v}{1+\frac{uv}{c^2}} \ .$$

One can compute Batman's γ-factor from this velocity, finding after some algebra that

$$\gamma_B = \gamma_u \gamma_v \left(1 + \frac{uv}{c^2}\right) \ .$$

For Robin's velocity the technique is the same, except u enters with the opposite sign. Thus

$$v_R^x = \frac{v-u}{1-\frac{uv}{c^2}}$$

$$\gamma_R = \gamma_u \gamma_v \left(1 - \frac{uv}{c^2}\right) \ .$$

One can now write down the answers quite easily. In this frame Batman will run for a time $\gamma_B \Delta t$, and will therefore travel a distance $\gamma_B v_B \Delta t$ before he jumps. With a similar expression for Robin, the separation between the two superheroes when they jump is given by

$$\text{Separation} = \gamma_B v_B \Delta t - \gamma_R v_R \Delta t = 2u\gamma_u\gamma_v \Delta t \ .$$

The time difference between the two jumps is given by

$$\text{Time Difference} = (\gamma_B - \gamma_R)\,\Delta t = \frac{2\gamma_u\gamma_v uv}{c^2} \Delta t \ .$$

Thus, the answers agree with the previous calculation.