MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01 Spring 2005

SOLUTIONS TO REVIEW PROBLEMS FOR FINAL EXAM

May 9, 2005

Problem 1 (Y&F:37.14): Inversion of the Lorentz Transformation

Solve Eq. (37.21),

$$\begin{aligned} x' &= \frac{1}{\sqrt{1-u^2/c^2}}(x-ut) = \gamma(x-ut) \\ y' &= y; \qquad z' = z \\ t' &= \frac{1}{\sqrt{1-u^2/c^2}}\left(t - \frac{ux}{c^2}\right) = \gamma\left(t-\frac{ux}{c^2}\right) , \end{aligned} \tag{37.21}$$

to obtain x and t in terms of x' and t', and show that the resulting transformation has the same form as the original one except for a change of sign for u.

Answer:

37.14: Multiplying the last equation of (37.21) by u and adding to the first to eliminate t gives

$$x' + ut' = \gamma x\left(1-\frac{u^2}{c^2}\right) = \frac{1}{\gamma}x,$$

and multiplying the first by $\frac{u}{c^2}$ and adding to the last to eliminate x gives

$$t' + \frac{u}{c^2}x' = \gamma t\left(1-\frac{u^2}{c^2}\right) = \frac{1}{\gamma}t,$$

so $x = \gamma(x' + ut')$ and $t = \gamma(t' + ux'/c^2)$,

which is indeed the same as Eq. (37.21) with the primed coordinates replacing the unprimed, and a change of sign of u.

Problem 2 (Y&F:37.20): Relativistic velocity addition

Two particles in a high-energy accelerator experiment are approaching each other head-on, each with a speed $0.9520c$ as measured in the laboratory. What is the magnitude of the velocity of one particle relative to the other?

Answer:

37.20: In the frame of one of the particles, u and v are both $0.9520c$ but with opposite sign.

$$v' = \frac{-v-(u)}{1-(u)\,(-v)/c^2} = \frac{-0.9520c - 0.9520c}{1-(0.9520)(-0.9520)} = -0.9988c.$$

Thus, one particle moves at a speed $0.9988c$ toward the other in the other particle's frame.

Problem 3 (Y&F:37.25): Tell it to the Judge

(a) How fast must you be approaching a red traffic light ($\lambda = 675$ nm) for it to appear yellow ($\lambda = 575$ nm)? Express your answer in terms of the speed of light.

(b) If you used this as an excuse for not getting a ticket for running a red light, how much of a fine would you get for speeding? Assume that the fine is $1.00 for each kilometer per hour that your speed exceeds the posted limit of 90 km/h.

Answer:

37.25: a) $f = \sqrt{\dfrac{c+u}{c-u}} f_0 \Rightarrow (c-u)f^2 = (c+u)f_0^2$

$$\Rightarrow u = \frac{c(f^2 - f_0^2)}{f_0^2 + f^2} = \frac{c((f/f_0)^2 - 1)}{(f/f_0)^2 + 1} = \frac{c((\lambda_0/\lambda)^2 - 1)}{((\lambda_0/\lambda)^2 + 1)}$$

$$\therefore u = c\frac{((675/575)^2 - 1)}{((675/575)^2 + 1)} = 0.159c = 4.77\times10^7 \text{ m/s} = 4.77\times10^4 \text{ km/s} = 1.72\times10^8 \text{ km/h.}$$

b) $(1.72\times10^8 \text{ km/h} - 90 \text{ km/h})\ (\$1.00) = \$172$ million dollars!

Problem 4 (Y&F:37.29): Relativistic momentum vs. Newtonian momentum

(a) At what speed is the momentum of a particle twice as great as the result obtained from the nonrelativistic expression mv? Express your answer in terms of the speed of light.

(b) A force is applied to a particle along its direction of motion. At what speed is the magnitude of force required to produce a given acceleration twice as great as the force required to produce the same acceleration when the particle is at rest? Express your answer in terms of the speed of light.

Answer:

37.29: a) $p = \dfrac{mv}{\sqrt{1 - v^2/c^2}} = 2mv$

$$\Rightarrow 1 = 2\sqrt{1 - v^2/c^2} \Rightarrow \frac{1}{4} = 1 - \frac{v^2}{c^2}$$

$$\Rightarrow v^2 = \frac{3}{4}c^2 \Rightarrow v = \frac{\sqrt{3}}{2}c = 0.866c.$$

b) $F = \gamma^3 ma = 2ma \Rightarrow \gamma^3 = 2 \Rightarrow \gamma = (2)^{1/3}$ so $\dfrac{1}{1 - \dfrac{v^2}{c^2}} = 2^{2/3} \Rightarrow \dfrac{v}{c}$

$$= \sqrt{1 - 2^{-2/3}} = 0.608.$$

Problem 5 (Y&F:37.37): Energy, momentum, and speed of a proton

A proton (rest mass 1.67×10^{-27} kg) has total energy that is 4.00 times its rest energy. What is

(a) the kinetic energy of the proton?

(b) the magnitude of the momentum of the proton?

(c) the speed of the proton?

Answer:

37.37: a) $E = mc^2 + K$, so $E = 4.00mc^2$ means $K = 3.00mc^2 = 4.50\times10^{-10}$ J

b) $E^2 = (mc^2)^2 + (pc)^2$; $E = 4.00mc^2$, so $15.0(mc^2)^2 = (pc)^2$

$p = \sqrt{15}mc = 1.94\times10^{-18}\,\text{kg}\cdot\text{m/s}$

c) $E = mc^2 \Big/ \sqrt{1 - v^2/c^2}$

$E = 4.00mc^2$ gives $1 - v^2/c^2 = 1/16$ and $v = \sqrt{15/16c} = 0.968c$

Problem 6 (Y&F:37.46): Energy of fusion

In a hypothetical nuclear-fusion reactor, two deuterium nuclei combine or "fuse" to form one helium nucleus. The mass of a deuterium nucleus, expressed in atomic mass units (u), is 2.0136 u; that of a helium nucleus is 4.0015 u (1 u = $1.6605402 \times 10^{-27}$ kg)

(a) How much energy is released when 1.0 kg of deuterium undergoes fusion?

(b) The annual consumption of electrical energy in the United States is of the order of 1.0×10^{19} J. How much deuterium must react to produce this much energy?

Answer:

37.46: a) The fraction of the initial mass that becomes energy is

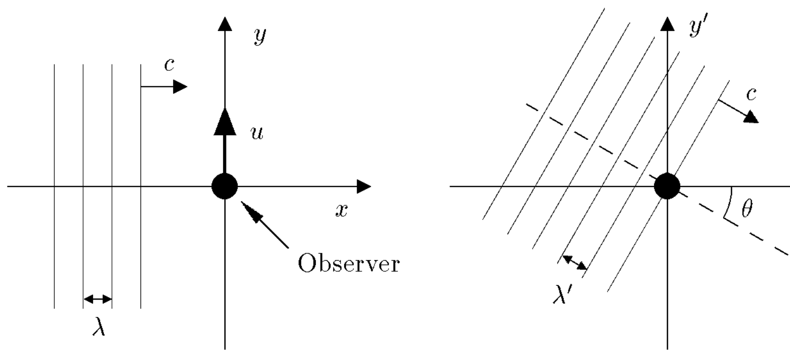
$1 - \dfrac{(4.0015\text{ u})}{2(2.0136\text{ u})} = 6.382\times10^{-3}$, and so the energy released per kilogram is

$(6.382\times10^{-3})(1.00\text{ kg})(3.00\times10^{8}\text{ m/s})^2 = 5.74\times10^{14}$ J.

b) $\dfrac{1.0\times10^{19}\text{ J}}{5.74\times10^{14}\text{J/kg}} = 1.7\times10^{4}$ kg.

Problem 7: Moving Perpendicular to a Light Wave

A planar light wave propagates along the x-axis, as shown on the diagram on the left. The wave moves of course at speed c, and it has a wavelength λ. An observer moves at speed u along the y-axis, as shown.



(a) Introduce a primed frame of reference which is at rest relative to the moving observer. Suppose, as we usually assume, that the origins of the two systems coincide at $t = t' = 0$. Write down the equations which give the primed coordinates in terms of the unprimed coordinates.

(b) In the primed reference frame, the wave propagates at an angle θ with respect to the x'-axis. Find θ.

(c) To the moving observer, the wavelength will appear Doppler shifted. Determine the wavelength λ' as seen by the moving observer. *(Caution: since the motion is perpendicular to the direction of wave propagation, the formulas in Young and Freedman do not apply. You must work from first principles.)*

Answer:

(a) This is a Lorentz transformation, with the velocity in the y-direction. So

$$\begin{aligned} t' &= \gamma\left(t - \frac{uy}{c^2}\right) \\ y' &= \gamma(y - ut) \\ x' &= x \\ z' &= z\ , \end{aligned}$$

where

$$\gamma = \frac{1}{\sqrt{1 - u^2/c^2}}\ .$$

(b) Consider a wavefront which crosses the y-axis at time $t = 0$, as shown at the top of the next page. The equations describing the locus of this wavefront are given by

$$x = ct\ , \qquad y = \text{anything}.$$

To determine the direction of propagation in the primed frame, we can calculate a snapshot of the wavefront at $t' = 0$. From the Lorentz transformation in (a),

$$t' = 0 \quad \Longrightarrow \quad t = \frac{uy}{c^2} \ .$$

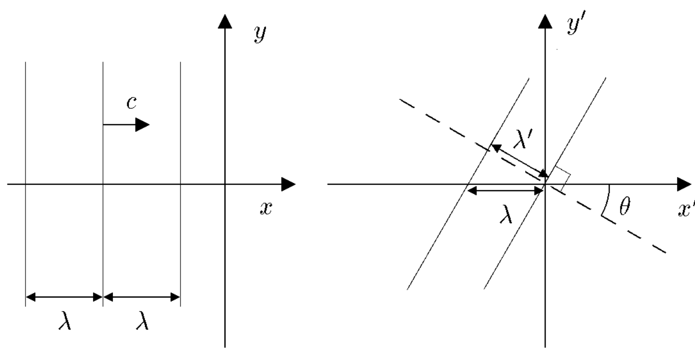
Transforming the x and y coordinates,

$$y' = \gamma(y - ut) = \gamma\left(y - \frac{u^2}{c^2}y\right) = y/\gamma$$

$$x' = x = ct = \frac{uy}{c} \ .$$

Then from trigonometry,

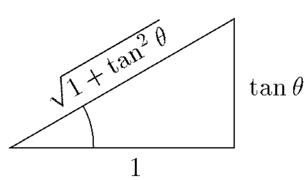
$$\boxed{\theta = \arctan\left(\frac{x'}{y'}\right) = \arctan\left(\frac{\gamma u}{c}\right) \ .}$$



(c) Since the x-coordinate is not changed by the transformation, the two wavefronts are separated along the x'-axis by a distance λ. As can be seen from the diagram, however, this distance is not the wavelength λ'— the wavelength is the separation between the wavefronts, measured along a line in the direction of propagation. By trigonometry,

$$\lambda' = \lambda \cos\theta \ .$$

From the figure



one can reconstruct the trigonometric identity

$$\cos\theta = \frac{1}{\sqrt{1+\tan^2\theta}} \ .$$

Using the answer to part (b), one finds with a little algebra that

$$\cos\theta = 1/\gamma \ .$$

Finally,

$$\boxed{\lambda' = \lambda/\gamma \ .}$$

Thus the wavelength is shortened, or blueshifted, by a factor γ.

[Alternatively, one could have derived the above result by noting that from the point of view of the unprimed frame, the wavefronts would reach the observer at intervals $\Delta t = \lambda/c$, just as if the observer were not moving. In this frame, however, the observers clocks are running slowly by a factor of γ. Thus the time interval as measured on the observer's clock is shortened by a factor γ, and therefore so is the wavelength as measured by the observer.]

Problem 8: Threshold for Particle Production

An electron of energy E (in the laboratory frame) collides with a proton at rest. Note that E denotes the ***total*** energy of the electron, which means kinetic energy plus rest energy. Let m_e denote the mass of the electron and let m_p denote the mass of the proton. In terms of m_e, m_p, and c, what is the minimum value of E that would allow the reaction

$$e^- + p \rightarrow e^- + p + p + \bar{p}$$

to take place? Here $\bar{p}$ denotes an antiproton, which also has a mass m_p. *(Hint: when E is at its minimum, the four particles of the final state will all move together, as if they were one particle of mass $m_e + 3m_p$. Any motion of these particles relative to each other would require a larger value of E.)*

Answer:

The key relationship needed for this problem is the relativistic relation between energy, momentum, and rest mass. Using $p^0 = E/c$ and $p^2 = -m^2c^2$, one has

$$E^2 - |\vec{p}|^2c^2 = M^2c^4 \ .$$

Thus the total energy and momentum of the initial state are given by

$$E_{\text{total}} = E + m_pc^2$$

$$|\vec{p}_{\text{total}}| = \sqrt{\frac{E^2}{c^2} - m_e^2c^2} \ .$$

The final state must of course have the same total energy and momentum, and will behave as if it were a particle of mass $m_e + 3m_p$. Thus,

$$E_{\text{total}}^2 - |\vec{p}_{\text{total}}|^2 c^2 = (m_e + 3m_p)^2 c^4 \ .$$

So

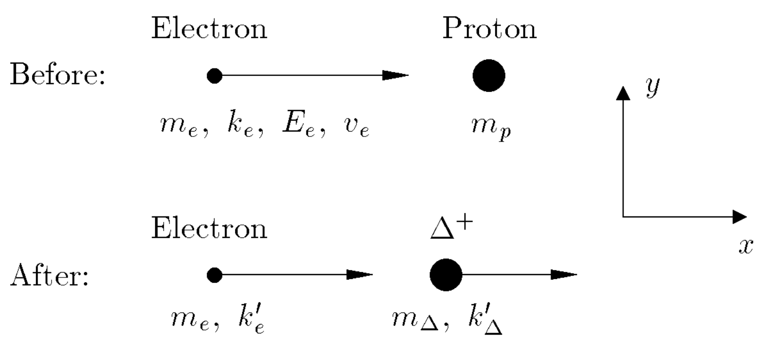
$$E^2 + 2Em_pc^2 + m_p^2c^4 - (E^2 - m_e^2c^4) = (m_e^2 + 6m_em_p + 9m_p^2)c^4 \ .$$

And finally,

$$\boxed{E = (4m_p + 3m_e)c^2 \ .}$$

Problem 9: A Relativistic Inelastic Collision

An electron of mass m_e and momentum k_e is incident on a stationary proton, of mass m_p. The electron moves along the x-axis, in the positive direction.



(a) What is the total energy E_e of the electron?

(b) What is the speed v_e of the electron?

(c) In the course of the collision the proton is excited, becoming a very short-lived particle (lifetime $\approx 10^{-23}$ sec.) called a Δ^+, with a mass m_Δ. Assume for simplicity that the final electron and Δ^+ continue to move along the x-axis, so it remains a one-dimensional problem. Denote the final momentum of the electron by k_e', and the final momentum of the Δ by k_Δ'. Write down two conservation equations that would allow one to solve for the two unknowns, k_e' and k_Δ'. You need not solve these equations.

Answer:

(a) The relativistic expression for the energy gives,

$$\boxed{E_e = \sqrt{k_e^2 c^2 + m_e^2 c^4} \quad .}$$

(b) The energy can be written equivalently as $E_e = \gamma_e m_e c^2$. Similarly one can write the relativistic momentum as $k_e = \gamma_e m_e v_e$. The velocity v_e is then found in terms of known quantities by dividing these equations, so that $v_e = k_e c^2 / E_e$ Substituting in the expression for E_e found in part (a), gives $v_e = k_e c^2 / \sqrt{k_e^2 c^2 + m_e^2 c^4}$, or equivalently

$$\boxed{v_e = \left(\frac{1}{\sqrt{1 + (m_e c/k_e)^2}} \right) c \quad .}$$

(c) The conservation equations are the conservation of relativistic energy and momentum. The conservation of energy equation reads $E_e + E_p = E_\Delta + E'_e$, or,

$$\boxed{\sqrt{k_e^2 c^2 + m_e^2 c^4} + m_p c^2 = \sqrt{k'^2_\Delta c^2 + m_\Delta^2 c^4} + \sqrt{k'^2_e c^2 + m_e^2 c^4} \quad .}$$

While the conservation of momentum gives

$$\boxed{k_e = k'_e + k'_\Delta \quad .}$$