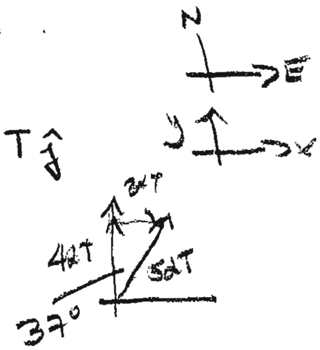
**Problem 1.** An object with zero velocity at time $t = 0$ accelerates Eastward with acceleration $3\alpha$ and Northward with acceleration $4\alpha$, where $\alpha$ is a constant. At time $t = T$ the object stops accelerating Eastward. Thereafter it accelerates Westward with acceleration $3\alpha$, with its Northward acceleration continuing unchanged. *It you choose to use angles, the following might prove useful:* $\cos^{-1}\frac{4}{5} = \sin^{-1}\frac{3}{5} = \tan^{-1}\frac{3}{4} \approx 37°$.

**a) What is the velocity at time $t = T$?**

$\vec{a} = 3\alpha\,\hat{\imath} + 4\alpha\,\hat{\jmath}$

$\vec{v} = \vec{a}t + \vec{v}_0 \; {}_{\to 0}$

$\vec{v}_T = 3\alpha T\,\hat{\imath} + 4\alpha T\,\hat{\jmath}$




COMMENTS

constant acceleration formula

let $x \Leftrightarrow E$
$y \Leftrightarrow N$

**b) What is the average acceleration between times $t = 0$ and $t = T$?**

acceleration is constant therefore average is initial value. Alternatively

$\langle \vec{a} \rangle = \dfrac{\vec{v}_f - \vec{v}_0}{T} = \dfrac{\vec{v}_F}{T} =$

$= \dfrac{3\alpha T}{T}\,\hat{\imath} + \dfrac{4\alpha T}{T}\,\hat{\jmath}$

$= 3\alpha\,\hat{\imath} + 4\alpha\,\hat{\jmath}$

COMMENTS

definition of average acceleration

**c) What is the position at time $t = T$?**

$\vec{r} = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}_{const}\, t^2$

$x = \cancel{x_0}^{\,0} + \cancel{v_{0x}}^{\,0} t + \frac{1}{2} 3\alpha t^2$

$y = \cancel{y_0}^{\,0} + \cancel{v_{0y}}^{\,0} t + \frac{1}{2} 4\alpha t^2$

$x_T = \frac{3}{2}\alpha T^2$

$y_T = 2\alpha T^2$

$\vec{r} = \frac{3}{2}\alpha T^2\,\hat{\imath} + 2\alpha T^2\,\hat{\jmath}$

COMMENTS

constant acceleration formula

let $\vec{v} = v_{0x}\,\hat{\imath} + v_{0y}\,\hat{\jmath}$

assume $x_0 = 0$
$y_0 = 0$

**d) What is the velocity at time $t = 2T$?**

$\vec{v} = \vec{a}(t-T) + \vec{v}_T$ minus!

$\vec{v} = (-3\alpha\hat{\imath} + 4\alpha\hat{\jmath})(2T-T) + 3\alpha T\hat{\imath} + 4\alpha T\hat{\jmath}$

$= 0\,\hat{\imath} + 8\alpha T\,\hat{\jmath}$

COMMENTS

constant acceleration formulae

Note that deceleration from T to 2T = acceleration from 0 to T

**e) What is the average acceleration between times $t = 0$ and $t = 2T$?**

$\langle \vec{a} \rangle = \dfrac{\vec{v}_f - \vec{v}_i}{2T}$

$\langle \vec{a} \rangle = \dfrac{8\alpha T\hat{\jmath}}{2T} = 4\alpha\,\hat{\jmath}$

COMMENTS

definition of average acceleration

**f) What is the position at time $t = 2T$?**

$y = \cancel{y_0}^{0} + \cancel{v_0 t}^{0} + \frac{1}{2} a_{0y} t^2$

$= \frac{1}{2} 4\alpha (2T)^2 = 8\alpha T^2$

$x = x_T + v_{Tx}(t-T) + \frac{1}{2} a_x (t-T)^2$

$x_{2T} = \underbrace{\frac{3}{2}\alpha T^2 + 3\alpha T(2T-T) - \frac{1}{2}3\alpha(2T-T)^2}_{\text{these terms cancel}}$

$x_{2T} = 3\alpha T^2$

COMMENTS

do y first, as easier constant acceleration formulae

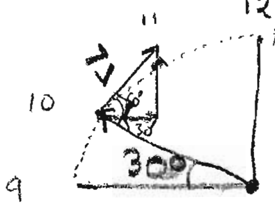
we know $x_T$; use constant accel formula taking initial position at $t=T$

**Problem 2.** **The second hand on a clock is $\frac{3}{2\pi}$ meters long and moves in uniform circular motion. Take the y-axis to be in the direction of 12 o'clock and the x-axis to be in the direction of 3 o'clock.** *You may leave your answer in terms of sines and cosines of specified angles or work through them if you like.*

**a) Calculate the** ***instantaneous*** **velocity of the tip of the second hand at 12:00:50.**

$$|\vec{v}| = \frac{\text{distance}}{\text{time}} = \frac{2\pi r}{T} \qquad r = \frac{3}{2\pi}\,\text{m} \qquad T = 60\,\text{s}$$

$$|\vec{v}| = \frac{2\pi\left(\frac{3}{2\pi}\right)\text{m}}{60\ \text{s}} = \frac{1}{20}\ \frac{\text{m}}{\text{s}}$$




$$\boxed{\vec{v} = |v|\left[\cos 60^\circ\,\hat{i} + \sin 60^\circ\,\hat{j}\right] = \frac{1}{20}\left[\frac{1}{2}\,\hat{i} + \frac{\sqrt{3}}{2}\,\hat{j}\right]\frac{\text{m}}{\text{s}}}$$

**COMMENTS**

GOAL: FIND INSTANTANEOUS $\vec{V}$

- MAGNITUDE IS CONSTANT
- DIRECTION IS TANGENT

$\Rightarrow$ $V_x > 0$, $V_y > 0$, $V_y > V_x$

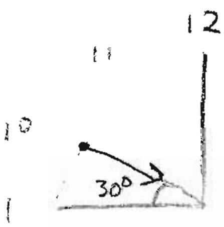
i.e. MORE UP THAN OVER

RESULT IS A VECTOR

**b) Calculate the** ***instantaneous*** **acceleration of the tip of the second hand at 12:00:50.** -GIVEN

GOAL

$$|\vec{a}| = \frac{|v|^2}{R} = \left(\frac{1}{20}\ \frac{\text{m}}{\text{s}}\right)^2 \cdot \left(\frac{2\pi}{3}\right)\frac{1}{\text{m}} = \frac{\pi}{600}\ \frac{\text{m}}{\text{s}^2}$$




$$\vec{a} = |a|\left[\cos 30^\circ\,\hat{i} - \sin 30^\circ\,\hat{j}\right]$$

$$= \frac{\pi}{600}\left[\frac{\sqrt{3}}{2}\,\hat{i} - \frac{1}{2}\,\hat{j}\right]\frac{\text{m}}{\text{s}^2}$$

**COMMENTS**

- MAGNITUDE IS CONSTANT GIVEN BY $|V|^2/R$
- DIRECTION ALWAYS INWARD

$a_x > 0$, $a_y < 0$

$|a_x| > |a_y|$

RESULT IS A VECTOR

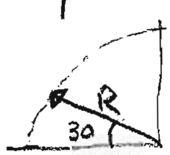
**c) Calculate the** ***average*** **velocity of the tip of the second hand between 12:00:50 and 12:01:00.**

GOAL GIVEN GIVEN

$\vec{X}_F$ : 12:01:00


$\vec{X}_F = R\,\hat{j}$

$\vec{X}_i$ : 12:00:50




$\vec{X}_i = R\left[-\cos 30^\circ\,\hat{i} + \sin 30^\circ\,\hat{j}\right]$

$\Delta T = 10\ \text{s}$

$$\langle\vec{v}\rangle = \frac{R\,\hat{j} - R\left[-\cos 30\,\hat{i} + \sin 30^\circ\,\hat{j}\right]\text{m}}{10\,\text{s}}$$

$$= \frac{R}{10}\left[\cos 30\,\hat{i} + (1 - \sin 30^\circ)\,\hat{j}\right]$$

$$= \frac{3}{20\pi}\ \frac{\text{m}}{\text{s}}\left[\frac{\sqrt{3}}{2}\,\hat{i} + \frac{1}{2}\,\hat{j}\right]$$

**COMMENTS**

DEFINITION $\langle\vec{V}\rangle = \dfrac{\vec{X}_f - \vec{X}_i}{\Delta T}$

$\vec{V}_i$ $\vec{V}_f$

MAKES SENSE THAT $\langle\vec{V}\rangle$ POINTS

RESULT IS A VECTOR

d) Calculate the *average* velocity of the tip of the second hand between 12:00:00 and 12:00:50.

$$\langle \vec{v} \rangle = \frac{\vec{X}_{12:00:50} - \vec{X}_{12:00:00}}{\Delta T}$$

$$= \frac{\vec{X}_{12:00:50} - \vec{X}_{12:01:00}}{\Delta T}$$

$$= \frac{-R\hat{j} + R[-\cos 30\hat{i} + \sin 30\hat{j}]\,m}{50\,s}$$

$$= \frac{3}{100\pi}\left[-\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}\right]$$

**COMMENTS**

- COULD PROCEDE AS IN c) BUT SINCE $\langle \vec{v} \rangle_{12:01-12:00} = 0$ CAN USE TRICK THAT $\vec{X}(12:00) = \vec{X}(12:01)$
- THIS IS $-\Delta\vec{R}$ (part c)
- $\Delta T = 50\,s$ IN THIS CASE
- RESULT IS A VECTOR

e) Calculate the *average* acceleration of the tip of the second hand between 12:00:00 and 12:00:50.

GIVEN GOAL

$$\vec{V}_i = |v|\,\hat{i}$$

$$\vec{V}_f = |v|\left[\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right]$$

$$\Delta T = 50\,s$$

$$\boxed{\langle \vec{a} \rangle = \frac{|v|\frac{m}{s}}{50\,s}\left[\left(\frac{1}{2} - 1\right)\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right] = \frac{1}{1000}\frac{m}{s^2}\left[-\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right]}$$

**COMMENTS**

DEFINITION $\langle \vec{a} \rangle = \dfrac{\vec{V}_f - \vec{V}_i}{\Delta T}$

$\vec{V}_f$ FROM PART a)

$\vec{V}_i$ HORIZONTAL

→ RESULT IS A VECTOR

f) Now consider the *average* acceleration of tip of the second hand between 12:00:50 and 12:01:00. What is the ratio of the magnitude of this acceleration to the magnitude of the acceleration calcultate in part e)?

$$\langle \vec{a} \rangle_f = \frac{\vec{V}(12:01) - \vec{V}(12:00:50)}{\Delta T_f}$$

$$= \frac{-\left(\vec{V}(12:00:50) - \vec{V}(12:00:00)\right)}{\Delta T_f}$$

$$= \frac{-\langle \vec{a} \rangle_e \cdot \Delta T_e}{\Delta T_f}$$

$$\boxed{\frac{|\langle a \rangle|_f}{|\langle a \rangle|_e} = \frac{\Delta T_e}{\Delta T_f} = \frac{50}{10} = 5}$$

**COMMENTS**

CAN RECALCULATE AND TAKE RATIO, BUT AGAIN B/C $\vec{V}(12:01) = \vec{V}(12:00)$

THAT MEANS $\Delta\vec{V}$ (PART f) $= -\Delta\vec{V}$ (PART e)

BUT $\Delta T_e$ IS 50 s, $\Delta T_f$ IS 10

SO THE AVERAGES WILL BE DIFFERENT

ANOTHER WAY OF SAYING THIS IS SINCE $\langle \vec{v} \rangle = 0$ AROUND A WHOLE CYCLE, IN f YOU GO THROUGH THE SAME $\Delta V$ IN $\frac{1}{5}$ THE TIME

Problem 3. A ball is launched from the ground at a 45 degree angle above the horizontal (which we take to be the x-axis). It rises to a maximum height, falls back down to the ground and bounces. After the bounce its velocity again makes a 45° angle to the horizontal x-axis, but the magnitude of the velocity right after the bounce is a factor of $\frac{3}{4}$ times the magnitude before the bounce. Letting the time of the launch be $t = 0$, the bounce occurs at $t = T$. After the bounce the ball rises to a new maximum height and then bounces a second time. You can ignore air resistance. Take the acceleration of gravity to be $g$.

a) What is the ball's maximum height between launch and the first bounce?

COMMENTS

$$h = h_0 + V_{0y} t - \frac{g t^2}{2}$$

$$V_y = V_{0y} - g t$$

$$0 = V_{0y} - g\frac{T}{2} \Rightarrow V_{0y} = \frac{gT}{2} \quad (*)$$

$$h_{max} = \left(g\frac{T}{2}\right)\frac{T}{2} - \frac{g}{2}\left(\frac{T}{2}\right)^2 = \frac{gT^2}{8}$$

at max height, at $\frac{T}{2}$, $V_y = 0$

$h_0 = 0$

b) What is the magnitude of the ball's initial velocity?

COMMENTS

$$V_{0y} = g\frac{T}{2}$$

$$V_0 = V_{0y}/\sin\theta = \frac{\sqrt{2}\, gT}{2}$$

see (*) in a)

c) How far does the ball travel between launch and the first bounce?

COMMENTS

$$d = V_{0x} \cdot T = V_0 \cos\theta\, T =$$

$$g\frac{T}{2}\sqrt{2}\,\frac{1}{\sqrt{2}} \cdot T = \frac{gT^2}{2}$$

x motion has constant velocity

d) By what factor is the time between the first and second bounces smaller than or larger than the time between launch and the first bounce?

COMMENTS

$$V_0 = \frac{\sqrt{2}\, gT}{2} \Rightarrow T = V_0 \frac{\sqrt{2}}{g}.$$

$$\therefore\ T_1 = V_1 \frac{\sqrt{2}}{g} = \frac{3}{4} T$$

$$T_1 = \frac{3}{4} T$$

according to part b)

After first bounce $V_1 = \frac{3}{4} V_0$, But trajectory ~~everything else~~ is the same

e) By what factor is the maxiumum height between the first and second bounces smaller than or larger than the maxiumum height between launch and the first bounce?

COMMENTS

$$h_{max} = \frac{gT^2}{8}$$

$$h_{max1} = \frac{gT_1^2}{8} = \left(\frac{3}{4}\right)^2 \frac{gT^2}{8} =$$

$$= \left(\frac{3}{4}\right)^2 h_{max}$$

according to part a.

trajectory (~~everything~~ is the same

f) By what factor is the distance between the first and second bounces smaller than or larger than the distance between launch and the first bounce?

COMMENTS

$$d = \frac{gT^2}{2}$$

$$d_1 = \frac{gT_1^2}{2} = \left(\frac{3}{4}\right)^2 d.$$

similar to e)