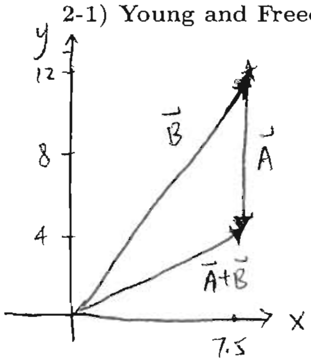
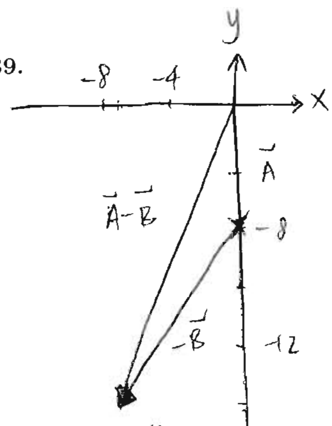
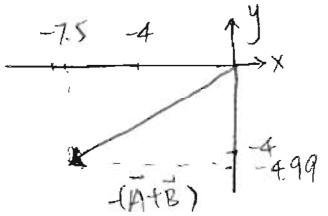
Physics 8.011 Pset #2 – Solutions

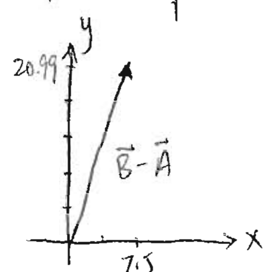
September 20, 2009

2-1) Young and Freedman 1.32, 1.39.









For the algebraic method, we decompose the vectors along the x and y axes:
$\vec{A} = (-8.00\,\text{m})\hat{y}$
$\vec{B} = (15.0\,\text{m} * \sin(30.0°))\hat{x} + (15.0\,\text{m} * \cos(30.0°))\hat{y}$
$= (7.5\,\text{m})\hat{x} + (12.99\,\text{m})\hat{y}$

a. $\vec{A}+\vec{B} = (7.5\,\text{m})\hat{x} + (-8.00 + 12.99\,\text{m})\hat{y}$, such that $\vec{A}+\vec{B} = (7.5\,\text{m})\hat{x} + (4.99\,\text{m})\hat{y}$. This gives $|\vec{A}+\vec{B}| = 9.01$ m at an angle of 33.6° from the x-axis.

b. $\vec{A}-\vec{B} = (-7.5\,\text{m})\hat{x} + (-8.00 - 12.99\,\text{m})\hat{y}$. such that $\vec{A}-\vec{B} = (-7.5\,\text{m})\hat{x} + (-20.99\,\text{m})\hat{y}$. Thus $|\vec{A}-\vec{B}| = 22.29$ m at an angle of 250.3° from the x-axis.

c. $|-\vec{A}-\vec{B}| = |\vec{A}+\vec{B}| = 9.01$ m and $-\vec{A}-\vec{B}$ is at $33.6° + 180° = 213°$ from the x-axis.

d. $|\vec{B}-\vec{A}| = |\vec{A}-\vec{B}| = 22.29$ m and $\vec{B}-\vec{A}$ is at $250.3° - 180° = 70.3°$ from the x-axis.

Commentary

Goal: To investigate vector addition and subtraction through algebraic and graphical methods.

In the graphical addition of vectors, the tail of the second vector is placed at the head of the first and the resulting vector is drawn from the free tail to the free head.

Subtracting a vector is equivalent to adding the negative of that vector, $(-\vec{V})$. The negative of a vector is obtained by flipping its head and tail.

The algebraic method provides a quantitative check to our drawing.

As before, the negative of a vector retains its magnitude and simply has its sign flipped, which is equivalent to a rotation by 180° in either direction.

Problem 2.2. Young and Freedman 1.51

a. *Is the vector $\hat{\imath} + \hat{\jmath} + \hat{k}$ a unit vector? Justify your answer.*

No, $\vec{r} = \hat{\imath} + \hat{\jmath} + \hat{k}$ is not a unit vector, since the magnitude $\sqrt{\vec{r} \cdot \vec{r}}$ is $\sqrt{(\hat{\imath} + \hat{\jmath} + \hat{k}) \cdot (\hat{\imath} + \hat{\jmath} + \hat{k})} = \sqrt{\hat{\imath} \cdot \hat{\imath} + \hat{\jmath} \cdot \hat{\jmath} + \hat{k} \cdot \hat{k}} = \sqrt{3}$. (The cross terms vanish, as $\hat{\imath}, \hat{\jmath}$, and $\hat{k}$ are orthogonal.) A unit vector has unit magnitude.

b. *Can a unit vector have any components with magnitude greater than unity? Can it have any negative components? In each case justify your answer.*

A unit vector must have unit magnitude. The magnitude of a vector is greater than or equal to the magnitude of any of its components. Therefore, a unit vector may not have any components with magnitude greater than unity.

However, a unit vector may have negative components, since a vector with negative components may still have unit magnitude. For instance, $-\hat{\imath} \cdot -\hat{\imath} = 1$.

(Consider the set $\left\{\hat{\jmath}, -\hat{\imath}, \hat{k}\right\}$. This is our usual set $\left\{\hat{\imath}, \hat{\jmath}, \hat{k}\right\}$ rotated 90° around the z axis, and constitutes a perfectly acceptable basis to describe $\Re^3$.)

c. *If $\vec{A} = a(3.0\hat{\imath} + 4.0\hat{\jmath})$, where a is a constant, determine the value of a that makes $\vec{A}$ a unit vector.*

We have $\sqrt{\vec{A} \cdot \vec{A}} = 1$, so $\vec{A} \cdot \vec{A} = 1$. We therefore have $a^2(3.0\hat{\imath} + 4.0\hat{\jmath}) \cdot (3.0\hat{\imath} + 4.0\hat{\jmath}) = 1$, or $25a^2 = 1$. Therefore, $a = \pm 5$. We could have gotten this more quickly by noting that the two components create a $3-4-5$ right triangle, so their sum – the hypotenuse of the triangle – has length 5.

Commentary

Goal: To identify and define unit vectors.

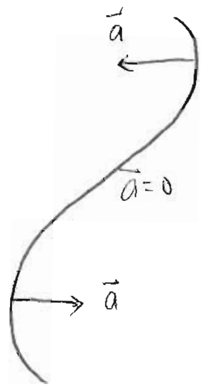
A unit vector is a whose magnitude (length) is one. A unit vector must also be, in a unfortunate turn of phrase, unitless; that is, it may not carry units like m/s, V/cm, etc. In this solution, we use the first criterion to help identify properties of unit vectors.

Any vector $\vec{r}$ in our vector space may be described in terms of unit vectors, though $\vec{r}$ is independent of our choice of unit vectors and its unit vector decomposition will depend on that (arbitrary) choice. However, some choices may be more judicious than others; for instance, in an external field, such as gravity, we probably would want to choose unit vectors such that one points along the field.

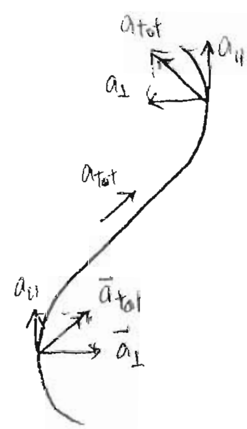
In some curvilinear coordinate systems (e.g. cylindrical, spherical), unit vectors may vary over space. Note that a may be either positive or negative.

Problem 2.3. Young and Freedman 3.8

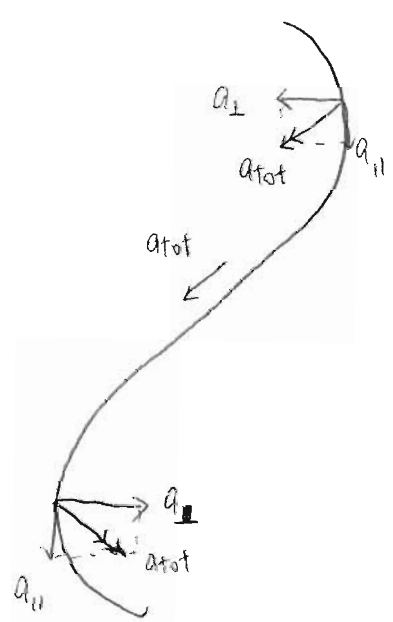
a.



b.



c.



Commentary

Goal: To find the acceleration vector from examining changes in magnitude and direction of the velocity.

With the velocity remaining constant, acceleration is directed simply towards the center of curvature of the trajectory.

With the velocity changing, now we need to add two separate and perpendicular components of acceleration: one perpendicular to the line of motion that changes the direction of the velocity, and the other along the line of motion that speeds up the particle. The net acceleration is found by graphically adding these two components of the vector.

In the last case, the acceleration along the line of motion is simply in the opposite direction as it slows down the particle.

Problem 2.4. Young and Freedman 3.15

Inside a starship at rest on the earth, a ball rolls off the top of a horizontal table and lands a distance D from the foot of the table. The starship now lands on the unexplored Planet X. The commander, Captain Curious, rolls the same ball off the same table with the same initial speed as on earth and finds that it lands a distance 2.76D from the foot of the table. What is the acceleration due to gravity on Planet X?

The quantities on Planet X are denoted with a prime, e.g. t'.

$$v'_x = v_x \tag{1}$$

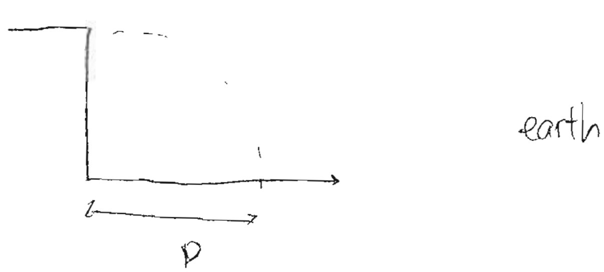
$$\frac{2.76D}{t'} = \frac{D}{t} \tag{2}$$

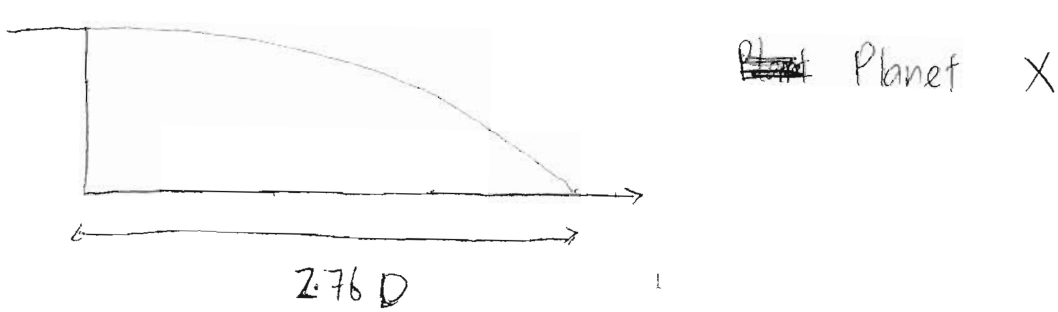
$$t' = 2.76t \tag{3}$$

Height of table $= \frac{1}{2}at^2$, and since the height of the table remains the same in both cases.

$$a't'^2 = at^2 \tag{4}$$

$$a' = a * \frac{t^2}{t'^2} = 9.81 * \frac{1}{2.76^2} = 1.29 m/s^2 \tag{5}$$





Commentary

Goal: To relate the 2-D trajectory of a ball with constant initial conditions to the value of the gravitational acceleration it experiences, by considering considering both horizontal and vertical motion separately.

Since the ball rolls horizontally off the table in each case, it has the same horizontal velocity in both cases. Since it lands further away in the second case, the time of flight must be longer in the second case.

In the constant gravitational field that we have on both planets. we can use the typical constant acceleration relation. Since the table is the same height on both planets, this drops out and we can relate the acceleration and time of flight on both planets. This gives us a relation we can solve to obtain the new acceleration on Planet X.

Problem 2.5

Consider a projectile fired horizontally from a cliff of given height. With what speed must it be fired so that it makes a 45° angle with the ground when it hits?

Consider the kinematic equation:

$$y = y_0 + v_{0,y}t + \frac{1}{2}at^2 \qquad (1)$$

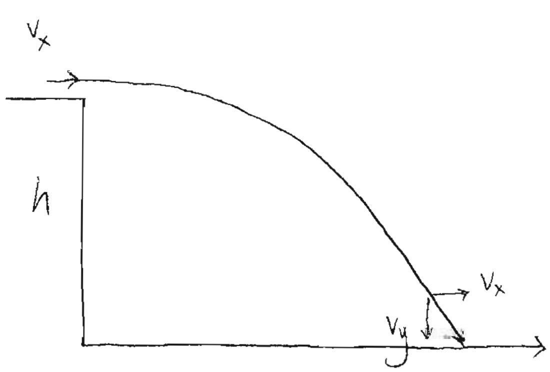
With $v_{0,y} = 0$ (the projectile is fired horizontally), $y_0 = h$ and $a = -g$, we have $\frac{1}{2}gt^2 = h$, where h is the height of the cliff. Rearranging, we have the now familiar time of impact $t = \sqrt{\frac{2h}{g}}$. Furthermore, we have another kinematic equation

$$v_y = v_{0,y} + at \qquad (2)$$

which gives $v_y = -gt$. Substituting the time t above, we have $v_y = -\sqrt{2gh}$.

From the commentary, we want the magnitude of v_x to be equal to that of v_y upon impact. With no horizontal acceleration, v_x is just $v_{0,x}$.

From the condition that the impact angle is 45°, we set $|v_{0,x}| = |v_x| = |v_y| = \sqrt{2gh}$.



Commentary

Goal: To infer the initial conditions of a projectile given the angle of impact with the ground.

Our first task is parsing the meaning of "making a 45° angle with the ground" upon impact. A projectile in a uniform gravitational field follows a parabolic trajectory. The angle must be the angle between the horizontal and the tangent of the projectile's path, determined where the path meets the horizontal. The tangent of a function is given by its derivative. If the trajectory is $\vec{r}(t)$, then the tangent is $\frac{d\vec{r}}{dt} = \vec{v}(t)$. $\vec{v}(t)$, then, must make a 45° angle with the horizontal. This condition is met when v_x and v_y have the same magnitude.

Figuring out this condition is actually the toughest part of this problem. Drawing the trajectory with a 45° impact angle is a helpful way to gain intuition for this condition.

Problem 2-6: Young/Freedman 3.28

On your first day at work for an appliance manufacturer, you are told to figure out what to do to the period of rotation during a washer spin cycle to triple the centripetal acceleration. You impress your boss by answering immediately. What do you tell her?

Assuming uniform circular motion, $a_c = \frac{v^2}{R}$, where R is the washing machine radius and v the tangential velocity. We want to relate v and R to the period τ. We know $\frac{v}{r} = \omega$, so we have $a_c = \omega^2 R$. Furthermore, $\omega = \frac{2\pi}{\tau}$, giving

$$a_c = \frac{4\pi^2 R}{\tau^2} \tag{1}$$

The new centripetal acceleration a_c' is $a_c' = 3a_c$, and we have a similar equation

$$3a_c = \frac{4\pi^2 R}{\tau'^2} \tag{2}$$

Dividing Equation 2 by Equation 1, we have

$$3 = \frac{\tau^2}{\tau'^2} \tag{3}$$

Rearranging, we have

$$\tau' = \frac{\tau}{\sqrt{3}} \tag{4}$$

Commentary

Goal: To investigate the relationship between centripetal acceleration and various parameters of uniform circular motion in order to meet new operation specifications of a washing machine.

The first task is to identify the various parameters in the problem: We have the tangential velocity v, the angular velocity ω, the period of motion τ, the washing machine radius R, and we want to triple a_c. v, ω, and τ are not independent, however: $\frac{v}{r} = \omega$, and $\omega = \frac{2\pi\,\mathrm{rad}}{\tau}$. Given these relationships, we can manipulate the familiar expression $a_c = \frac{v^2}{R}$ to give an expression relating a_c to τ and R.

We are told we are to triple a_c by adjusting the period. Writing a new expression for new period τ', we see that R drops out of the problem when you take the appropriate ratio. The result is an expression for τ' as τ times a numerical factor.

Problem 2.7

The second hand on a big clock is 1 meter long. Take the y-axis to be in the direction of 12 o'clock and the x-axis to be in the direction of 3 o'clock.

a. *Calculate the instantaneous velocity and acceleration of the second hand at i.) 12:00:00, ii.) 12:00:10, iii.) 12:00:15, iv.) 12:00:30 and v.) 12:01:00.*

b. *Calculate the average velocity and acceleration of the second hand between 12:00:00 and i.) 12:00:10, ii.) 12:00:15, iii.) 12:00:30, and iv.) 12:01:00.*

The magnitude of the instantaneous velocity v is $v = \omega R$, where $R = 1\,\text{m}$ and $\omega = \frac{2\pi\,\text{rad}}{60\,\text{s}}$. Therefore, $v = 0.105\,\text{m/s}$. Likewise, the magnitude of the instantaneous acceleration a_c is $a_c = \frac{v^2}{R}$, or $a_c = 0.011\,\text{m/s}^2$.

Let's define the angle θ to be that between the $\hat{j}$ direction and the second hand. Then the tip of the second hand is located at $\vec{r} = R\left[sin(\theta(t))\hat{i} + cos(\theta(t))\hat{j}\right]$ at a time t. Differentiating with the chain rule, we have $\vec{v} = R\frac{d\theta}{dt}\left[cos(\theta(t))\hat{i} - sin(\theta(t))\hat{j}\right]$. Since $\frac{d\theta}{dt} = \omega$ and $\omega R = v$, we have $\vec{v} = v\left[cos(\theta(t))\hat{i} - sin(\theta(t))\hat{j}\right]$. Differentating yet again, we have $\vec{a} = \omega v\left[-sin(\theta(t))\hat{i} - cos(\theta(t))\hat{j}\right]$, or $\vec{a} = -a_c\left[sin(\theta(t))\hat{i} + cos(\theta(t))\hat{j}\right]$.

For the instantaneous quantities, we only need to plug in values for $\theta(t)$. Taking t to be the amount of time, in seconds, after 12:00:00, we have $\theta(t) = 2\pi\frac{t}{60} = \frac{\pi t}{30}$. So

i.) $\vec{v} = (0.105\,\text{m/s})\hat{i}$, $\quad \vec{a} = (-0.011\,\text{m/s}^2)\hat{j}$
ii.) $\vec{v} = (0.105\,\text{m/s})\left[\frac{\sqrt{3}}{2}\hat{i} - \frac{1}{2}\hat{j}\right]$, $\quad \vec{a} = (-0.011\,\text{m/s}^2)\left[\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right]$
iii.) $\vec{v} = (-0.105\,\text{m/s})\hat{j}$, $\quad \vec{a} = (-0.011\,\text{m/s}^2)\hat{i}$
iv.) $\vec{v} = (-0.105\,\text{m/s})\hat{i}$, $\quad \vec{a} = (0.011\,\text{m/s}^2)\hat{j}$
v.) $\vec{v} = (0.105\,\text{m/s})\hat{i}$, $\quad \vec{a} = (-0.011\,\text{m/s}^2)\hat{j}$

For average velocities and accelerations, take $\langle\vec{v}\rangle = \frac{\vec{r}(t)-\vec{r}(0)}{t}$ and $\langle\vec{a}\rangle = \frac{\vec{v}(t)-\vec{v}(0)}{t}$. $\vec{r}(0) = (1\,\text{m})\hat{j}$ and $\vec{v}(0) = (0.105\,\text{m/s})\hat{i}$. Then

i.) $\langle\vec{v}\rangle = \left[(0.0866\,\text{m/s})\hat{i} + (-0.05\,\text{m/s})\hat{j}\right]$
$\quad\langle\vec{a}\rangle = \left[(-0.0052\,\text{m/s}^2)\hat{i} + (-0.0091\,\text{m/s}^2)\hat{j}\right]$
ii.) $\langle\vec{v}\rangle = \left[(0.0667\,\text{m/s})\hat{i} + (-0.0667\,\text{m/s})\hat{j}\right]$
$\quad\langle\vec{a}\rangle = \left[(-0.007\,\text{m/s}^2)\hat{i} + (-0.007\,\text{m/s}^2)\hat{j}\right]$
iii.) $\langle\vec{v}\rangle = (-0.0667\,\text{m/s})\hat{j}$, $\quad \langle\vec{a}\rangle = (-0.007\,\text{m/s}^2)\hat{i}$
iv.) $\langle\vec{v}\rangle = 0$, $\quad \langle\vec{a}\rangle = 0$.

Commentary

Goal: To investigate instantaneous and average velocities and accelerations in uniform circular motion.

As this is uniform circular motion, we know that *magnitude* of velocity v and *magnitude* of centripetal acceleration a_c is the same for all positions of the second hand. Let's calculate it first.

For direction, $\vec{v}$ is along the rim of the clock in the clockwise direction, and $\vec{a}$ points towards the center of the circle. To describe this more precisely, we can differentiate the vector $\vec{r}$ as follows.

We define a position vector $\vec{r}$, which is a function of θ, itself a function of time t. An quick check on our $\vec{r}$ is to plug in $\theta = 0$, $\theta = \pi/2$, and so on: these all give the correct values. Next, we differentiate $\vec{r}$ twice to obtain $\vec{v}$ and $\vec{a}$, finding the magnitudes are as calculated above. The quantities in square brackets are all unit vectors. (Note that if we did not have a constant ω, i.e. if our clock were losing or gaining time, we'd have to invoke the product rule when differentiating $\vec{v}$.)

We only need to determine values of θ and plug them in to our formulae to obtain instantaneous quantities. For average velocities and accelerations, we take $\langle\vec{v}\rangle = \frac{\vec{r}(t)-\vec{r}(0)}{t}$ and $\langle\vec{a}\rangle = \frac{\vec{v}(t)-\vec{v}(0)}{t}$.

Problem 2.8. Young and Freedman 3.35. Hypergravity At its Ames Research Center, NASA uses its large "20-G" centrifuge on test pilots and astronauts. In this device, an arm 8.84 m long rotates about one end in a horizontal plane, and the astronaut is strapped in at the other hand. Suppose that he is aligned along the arm with his head at the outermost end. The maximum sustained accelerated to which humans are subjected in this machine is typically $12.5g$.

a. *How fast must the astronaut's head be moving to experience this maximum acceleration?*

For uniform circular motion, we have $v^2/R = a_c$ for the centripetal acceleration. Rearranging, we have $v = \sqrt{Ra_c} = \sqrt{8.84\,\mathrm{m} * 12.5 * 9.8\,\mathrm{m/s^2}} = 32.9\,\mathrm{m/s}$.

b. *What is the difference between the acceleration of his feet if the astronaut is* 2.00 m *tall?*

We need to relate the two points at $r = 8.84$ m and $r = 6.84$ m. As noted in the commentary, we do so through $a = \omega^2 r$
Then

$$a_{head} = \omega^2 r_{head} \tag{1}$$

$$a_{feet} = \omega^2 r_{feet} \tag{2}$$

Taking Eq 1 - Eq 2. we get

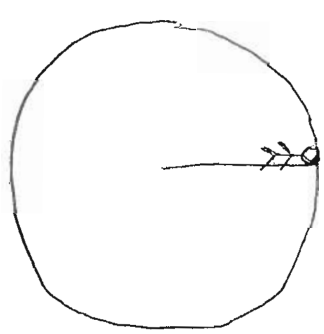
$$\begin{aligned}\Delta a &= \omega^2(r_{head} - r_{feet}) \\ &= \frac{a_{head}}{r_{head}} * (r_{head} - r_{feet}) \\ &= \frac{12.5g}{8.84m} * 2m \\ &= 2.83g\end{aligned}$$

c. *How fast in rpm (rev/min) is the arm turning to produce the maximum sustained acceleration?*

The angular velocity is $\omega = v_2/R_2 = 3.72\,\mathrm{rad/s}$. We need to convert this to rpm:

$$\omega = v_2/R_2 = 3.72\,\mathrm{rad/s} = 3.72\,\mathrm{rad/s}\frac{60\,\mathrm{s}}{1\,\mathrm{min}}\frac{1\,\mathrm{rev}}{2\pi\,\mathrm{rad}} \tag{3}$$

This gives $\omega = 35.5$ rpm.



Commentary

Goal: To investigate uniform circular motion and centripetal acceleration in a hypergravity device.

The first item is straightforward application of our equation for centripetal acceleration, $v^2/R = a_c$. The second item is a little tricky; we need to relate two points separated by a radial distance of 2.00 m. r is certainly not the same between them. What about v? It should also vary in the radial direction, since the center point at $r = 0$ should be at rest. What is the same between the two points is the angular velocity $\omega = v/R$.

Having found ω, we have essentially already solved Part C; we just need to convert units.

Problem 2.9

Data point	Time	Position	Displacement	Average Velocity	Change in velocity	Average acceleration	Instantaneous velocity	Instantaneous acceleration
1	0.06	1.37	0.00	0.00	0.00	0	0.00	0.00
2	0.12	1.32	0.05	0.42	0.42	6.97	0.84	13.94
3	0.18	1.23	0.09	0.78	0.36	6.04	1.50	11.16
4	0.24	1.10	0.13	1.13	0.35	5.81	2.17	11.16
5	0.30	0.91	0.19	1.54	0.41	6.83	3.17	16.73
6	0.36	0.71	0.20	1.84	0.30	5.02	3.34	2.79
7	0.42	0.50	0.21	2.08	0.24	3.98	3.51	2.79
8	0.48	0.23	0.27	2.38	0.30	5.08	4.51	16.73
9	0.54	-0.06	0.29	2.65	0.27	4.57	4.84	5.58

Average acceleration	10.11

Commentary

Goal: To find the acceleration of gravity from measurements of position and time.

We can calculate the time interval by taking the inverse of frequency. A table is created using the guidelines in the problem set. Data was taken from Class Photo 2.

Position is read off from the photo, and displacement is the difference in position travelled between each data point. I included instantaneous velocity by dividing the displacement over each time interval. Lastly, the instantaneous acceleration is found by taking the difference in each instantaneous velocity over each time interval, divided by the time interval, i.e. $\vec{a} = \frac{dv}{dt}$

Averaging the values obtained for the instantaneous acceleration gives a result of $10.11 m/s^2$. which is reasonably close to the accurate value of gravity given the roughness of data used.