IC_W04D2-1 Bead Moving Circularly Along Inside Surface of Cone Solution

A body of mass $m$ slides without friction on the inside of a cone. The axis of the cone is vertical and gravity is directed downwards. The apex half-angle of the cone is $\beta$ as shown in the figure. The path of the object happens to be a circle in a horizontal plane. The speed of the particle is $v_0$. Find the radius of the circular path and the time it takes to complete one circular orbit in terms of the given quantities and $g$.

**Solution:** The free body diagram of the forces acting on the object is shown in the figure below.

We choose cylindrical coordinates with unit vectors $\hat{r}$ pointing in the radial outward direction and $\hat{k}$ pointing upwards. The two forces acting on the object are the normal force $\vec{N}$ of the wall on the object and the gravitational force $mg$ (we are given that there is no friction). The acceleration is directed towards the center of the circular orbit as shown in the acceleration diagram below.

The two vector components of Newton’s Second Law $\vec{F} = m\vec{a}$ become:

$$\hat{r} : -N \cos \beta = \frac{-m v_0^2}{r}, \quad (0.1)$$

$$\hat{k} : N \sin \beta - mg = 0. \quad (0.2)$$
These equations can be re-expressed as

\[ N \cos \beta = m \frac{v_0^2}{r} , \]  
\[ N \sin \beta = m g . \]  

(0.3)  

(0.4)

We can divide these two equations,

\[ \frac{N \sin \beta}{N \cos \beta} = \frac{m g}{m \frac{v_0^2}{r}} , \]  
yielding

\[ \tan \beta = \frac{r g}{v_0^2} . \]  

(0.5)  

(0.6)

Equation (0.6) can be solved for the radius,

\[ r = \frac{v_0^2}{g} \tan \beta . \]  

(0.7)

Note that the radius is independent of the mass because the component of the normal force in the vertical direction must balance the gravitational force, and so the normal force is proportional to the mass. Therefore the radially inward component of the normal force is also proportional to mass. Since this force is causing the object to accelerate inward, by Newton’s Second Law, the mass will cancel and the acceleration is independent of the mass. Since the radial component of the acceleration is inversely proportional to the radius, \( a_r = -\frac{v_0^2}{r} \), therefore the radius is also independent of the mass. The period for the orbit is given by

\[ T = \frac{2\pi r}{v_0} = \frac{2\pi v_0 \tan \beta}{g} . \]  

(0.8)