IC-W06D2-1 Work Done by the Spring Force Solution

Connect one end of a spring of length $l_{eq}$ with spring constant $k$ to an object resting on a smooth table and fix the other end of the spring to a wall. Stretch the spring until it has length $l_0$ and release the object. How much work does the spring do on the object as a function of $x \equiv l - l_{eq}$, the distance the spring has been stretched or compressed?

**Solution:** We first begin by choosing a coordinate system with origin at the position of the body when the spring is at rest in the equilibrium position. We choose the $\hat{i}$ unit vector to point in the direction the body moves when the spring is being stretched. We choose the coordinate function $x(t)$ to denote the position of the body with respect to the origin (equilibrium position) at time $t$. In Figure 2 we illustrate the coordinate system by showing the equilibrium position and the position of the body at time $t$ when the spring is stretched, $x(t)$. Note that at $t = t_0$, the position of the body is $x_0 \equiv x(t = t_0) = l_0 - l_{eq}$, and at time $t$ the position of the body is denoted by $x(t) = l(t) - l_{eq}$. Note that $x_0$ and $x(t)$ can be positive, zero, or negative.

![Figure 2 Equilibrium position and position at time $t$](image)

The spring force on the body is given by

$$\vec{F} = F_x \hat{i} = -k x \hat{i}. $$
In Figure 3 we show the graph of the $x$-component of the spring force as a function of $x$ for positive values of $x$ corresponding to stretching of the spring.

![Graph of the $x$-component of the spring force as a function of $x$.]

**Figure 3** The $x$-component of the spring force as a function of $x$.

The work done is just the area under the curve for the interval $x_0$ to $x(t)$,

$$W = \int_{x=x_0}^{x=x(t)} F_x \, dx = \int_{x=x_0}^{x=x(t)} -k x \, dx .$$

This integral is straightforward and the work done by the spring force on the body is

$$W = \int_{x=x_0}^{x=x(t)} -k x \, dx = \frac{1}{2} k \left( x(t)^2 - x_0^2 \right) .$$

When the absolute value of $x(t)$ is less than the absolute value of the initial distance, $|x(t)| < |x_0|$, the work done is positive. This means that if the spring is less stretched or compressed at time $t$ than in the initial state, the work done by the spring force is positive. The spring force does positive work on the body when the spring goes from a state of “greater tension” to a state of “lesser tension.”