Problem 2 Sliding Along a Sphere Solution

An object of mass $m$ initially sits on top of a large sphere of radius $R$ that is fixed to the ground as shown in the figure. The object gets a small push and begins to slide along the surface of the sphere. You may assume that the initial kinetic energy is negligible. There is a friction force between the object and the surface that varies with the angle $\theta$ according to $f = f_0 \sin \theta$ where $f_0 < mg$ is a constant. Let $g$ denote the magnitude of acceleration due to gravity.

The object just loses contact with the surface of the sphere at an angle $\theta_f$ with respect to the vertical when it has a speed $v_f$.

a) What is the work done by the friction force on the object as the object moves through the angle $\theta = 0$ to the angle $\theta = \theta_f$? Hint: the small displacement of the object on the surface of the sphere is given by $d\mathbf{r} = R d\theta \hat{\theta}$.

Solution: The friction force is $\mathbf{f} = -f_0 \sin \theta \hat{\theta}$. The work done by the friction force is

$$W^f = \int_{\theta' = 0}^{\theta' = \theta_f} \mathbf{f} \cdot d\mathbf{r} = \int_{\theta' = 0}^{\theta' = \theta_f} -f_0 \sin \theta' \hat{\theta} \cdot R d\theta' \hat{\theta} = -f_0 R \int_{\theta' = 0}^{\theta' = \theta_f} \sin \theta' d\theta'$$

$$W^f = f_0 R \cos \theta' |_{\theta' = 0}^{\theta' = \theta_f} = f_0 R (\cos \theta_f - 1)$$
b) What is the work done by the gravitational force on the object as the object moves through the angle \( \theta = \theta_0 \) to the angle \( \theta = \theta_f \)?

Solution: The gravitational force is \( \mathbf{F}_g = mg \sin \theta \hat{\mathbf{r}} - mg \cos \theta \hat{\mathbf{r}} \). The work done by the gravitational force is

\[
W_g = \int_{\theta_0}^{\theta_f} (mg \sin \theta' \hat{\mathbf{r}} - mg \cos \theta' \hat{\mathbf{r}}) \cdot \mathbf{R} \, d\theta' = mg \int_{\theta_0}^{\theta_f} \sin \theta' \, d\theta'
\]

\[
W_g = -mgR \cos \theta \bigg|_{\theta_0}^{\theta_f} = -mgR(\cos \theta_f - 1)
\]

c) Based on your results from parts a) and b), use the work-energy theorem to determine an expression for the kinetic energy of the object just before the object leaves the surface of the sphere in terms of \( \theta_f \), \( m \), \( g \), \( R \), \( f_0 \), and \( v_f \) as needed.

Solution: The sum of the work is

\[
W = W_f + W_g = -mgR(\cos \theta_f - 1) + f_0 R(\cos \theta_f - 1) = R(1 - \cos \theta_f)(mg - f_0).
\]

The work-energy theorem is then

\[
R(1 - \cos \theta_f)(mg - f_0) = (1/2)mv_f^2
\]  

(1)

d) When the object reaches the angle \( \theta_f \), apply Newton’s Second Law to determine a second independent relationship between \( \theta_f \), \( m \), \( g \), \( R \), and \( v_f \) as needed.

Solution: when the object leaves the surface at \( \theta_f \), the normal force is zero, so Newton’s Second Law in the radial direction is

\[-mg \cos \theta_f = -mv_f^2 / R.\]

Thus

\[
\cos \theta_f = v_f^2 / gR
\]  

(2)

e) Using your results from parts c) and d), determine an expression for \( v_f \) in terms of \( m \), \( g \), \( R \), and \( f_0 \). Do not include \( \theta_f \) in your answer.
Solution: Substitute Eq. (2) into Eq. (1) yields

\[ R(1 - v_f^2 / gR)(mg - f_o) = (1/2)mv_f^2. \]

A little rearranging yields

\[ R(mg - f_o) = (1/2)mv_f^2 + (v_f^2 / g)(mg - f_o) = (v_f^2 / g)((3 / 2)mg - f_o) \]

We can now solve for the speed of the object when it just leaves the sphere

\[ v_f = \sqrt{Rg \frac{mg - f_o}{((3 / 2)mg - f_o)}}. \]