IC_W07D1-1 Table Problem: Change in Potential Energy Spring Force Solution

Connect one end of a spring of length $l_0$ with spring constant $k$ to an object resting on a smooth table and fix the other end of the spring to a wall. Stretch the spring until it has length $l$ and release the object. Consider the object-spring as the system. When the spring returns to its equilibrium length what is the change in potential energy of the system?

**Solution:** Consider a spring-body system lying on a frictionless horizontal surface with one end of the spring fixed to a wall and the other end attached to a body of mass $m$ (figure below). The spring force is an internal conservative force. The wall exerts an external force on the spring-body system but since the point of contact of the wall with the spring undergoes no displacement this external force does no work.

Choose the origin at the position of the center of the body when the spring is relaxed (the equilibrium position). Let $x$ be the displacement of the body from the origin. We choose the $+\hat{i}$ unit vector to point in the direction the body moves when the spring is being stretched (to the right of $x = 0$ in the figure). The spring force on a mass is then given by

$$\vec{F} = F_x \hat{i} = -kx \hat{i}. \quad (1)$$

The work done by the spring force on the mass is

$$W_{spring} = \int_{x=x_0}^{x=x_f} (-kx) \, dx = -\frac{1}{2}k(x_f^2 - x_0^2). \quad (2)$$
We then define the change in potential energy in the spring-body system in moving the body from an initial position \( x_0 \) from equilibrium to a final position \( x_f \) from equilibrium by

\[
\Delta U_{\text{spring}} \equiv U_{\text{spring}}(x_f) - U_{\text{spring}}(x_0) = -W_{\text{spring}} = \frac{1}{2} k(x_f^2 - x_0^2).
\] (3)

Therefore for a stretch of the spring-body system from equilibrium \( x_0 = 0 \) to a final position \( x_f = l - l_0 \), the potential energy changes by

\[
\Delta U = U_{\text{spring}}(x_f) - U_{\text{spring}}(x_0) = \frac{1}{2} k(l - l_0)^2.
\] (4)

Note: For the spring-body system, there is an obvious choice of position where the potential energy is zero, the equilibrium position of the spring-body,

\[
U_{\text{spring}}(x = 0) \equiv 0.
\] (5)

Then with this choice of zero reference potential, the potential energy function is given by

\[
U_{\text{spring}}(x) = \frac{1}{2} k x^2, \quad \text{with} \quad U_{\text{spring}}(x = 0) \equiv 0.
\] (6)