An object of mass $m$ is released from an initial state of rest from a spring of constant $k$ that has been compressed a distance $x_0$. After leaving the spring (at the position $x = 0$ when the spring is unstretched) the object travels a distance $d$ along a horizontal track that has a coefficient of friction that varies with position as

$$\mu = \mu_0 + \mu_1 \left( \frac{x}{d} \right).$$

Following the horizontal track, the object enters a quarter turn of a frictionless loop whose radius is $R$. Finally, after exiting the quarter turn of the loop the object travels vertically upward to a maximum height, $h$, (as measured from the horizontal surface). Let $g$ be the gravitational constant. Find the maximum height, $h$, that the object attains. Express all answers in terms of $m$, $k$, $x_0$, $g$, $\mu_0$, $\mu_1$, $d$ and $R$; not all variables may be needed.

**Solution:**

This problem may seem complicated at first (all those parameters!), but the work-energy theorem makes it tractable, even simple. Take the initial state to be when the spring is compressed the distance $x_0$ and the final state to be when the object is at its maximum height. The initial energy is then

$$E_{\text{initial}} = \frac{1}{2} k x_0^2$$

and the final energy is

$$E_{\text{final}} = mgh;$$

neither expression contains a kinetic energy term.

The nonconservative work is that done by friction. The magnitude of the friction force is $f = \mu mg$ and the nonconservative work is

$$W_{nc} = -\int f \, dx = -mg\int \mu \, dx. \quad (1)$$

If the coefficient of friction were constant ($\mu_1 = 0$), this wouldn’t be a difficult problem at all, and with the expression given, the integral is not at all hard. We have
\[ W_{nc} = -mg \int \mu \, dx \]
\[ = -mg \int_0^d \left( \mu_0 + \mu_1 \left( \frac{x}{d} \right) \right) dx \]
\[ = -mg \left( \mu_0 d + \mu_1 d/2 \right) \]
\[ = -mgd \left( \mu_0 + \mu_1 / 2 \right). \]  

It should be noted that in the figure, \( x \) increases from right to left. In any event, the non-conservative work done by friction is negative.

The work-energy theorem is then
\[ mgh - \frac{1}{2} k x_0^2 = -mgd \left( \mu_0 + \mu_1 / 2 \right), \]
which is easily solved for
\[ h = \frac{k x_0^2}{2mg} - d \left( \mu_0 + \mu_1 / 2 \right). \]

The result of Equation (4) must be qualified, in that if that result were negative, the friction would be enough to stop the object before it entered the loop. Also, there is no reason why \( \mu_1 \) could not be negative, which would mean the friction force decreases in magnitude with increasing \( x \). However, if this were the case, we would have to have \( \mu_0 + \mu_1 \geq 0 \).