A small block of mass \( m \) is pushed against a spring with spring constant \( k \) and held in place with a catch. The spring compresses an unknown distance \( x \). When the catch is removed, the block leaves the spring and slides along a frictionless circular loop of radius \( r \). When the block reaches the top of the loop, the force of the loop on the block (the normal force) is equal to twice the gravitational force on the mass. How far was the spring initially compressed?

**Solution:**

We will use conservation of energy to find the kinetic energy of the block at the top of the loop. We will then use Newton’s Second Law, to derive the equation of motion for the block when it is at the top of the loop. Specifically, we will find the speed \( v_{\text{top}} \) in terms of the gravitational constant \( g \) and the loop radius \( r \). We will then combine these results to find how far the spring was initially compressed.

**Choice of Zero for Potential Energy:** choose the gravitational potential energy to be zero at the bottom of loop.

**Initial Energy:** Choose for the initial state the instant before the catch is released. The initial kinetic energy is \( K_0 = 0 \). The initial potential energy is non-zero, \( U_0 = (1/2)k x^2 \). The initial mechanical energy is then

\[
E_0 = K_0 + U_0 = (1/2)k x^2. \tag{1}
\]

**Final Energy:** Choose for the final state the instant the block is at the top of the loop. The final kinetic energy is \( K_f = \frac{1}{2} m v_{\text{top}}^2 \); the mass is in motion with speed \( v_{\text{top}} \). The final potential energy is non-zero, \( U_f = (mg)(2R) \). The final mechanical energy is then
\[ E_f = K_f + U_f = 2mgR + \frac{1}{2}mv_{\text{top}}^2. \]  

(2)

Non-conservative Work: Since we are assuming the track is frictionless, there is no non-conservative work.

Change in Mechanical Energy: The change in mechanical energy is therefore zero,

\[ 0 = W_{\text{nc}} = \Delta E_{\text{mechanical}} = E_f - E_0. \]  

(3)

Mechanical energy is conserved, \( E_f = E_0 \), or

\[ 2mgR + \frac{1}{2}mv_{\text{top}}^2 = \frac{1}{2}kx^2. \]  

(4)

From Equation (4), the kinetic energy at the top of the loop is

\[ \frac{1}{2}mv_{\text{top}}^2 = \frac{1}{2}kx^2 - 2mgR. \]  

(5)

At the top of the loop, the forces on the block are the gravitational force of magnitude \( mg \) and the normal force of magnitude \( N \), both directed down. Newton’s Second Law in the radial direction, which is the downward direction, is

\[ -mg - N = -\frac{mv_{\text{top}}^2}{R}. \]  

(6)

In this problem, we are given that when the block reaches the top of the loop, the force of the loop on the block (the normal force, downward in this case) is equal to twice the magnitude gravitational force on the block, \( N = 2mg \). The Second Law, Equation (6), then becomes

\[ 3mg = \frac{mv_{\text{top}}^2}{R}. \]  

(7)

We can rewrite Equation (7) in terms of the kinetic energy as

\[ \frac{3}{2}mgR = \frac{1}{2}mv_{\text{top}}^2. \]  

(8)

Combining Equations (5) and (8) yields
\[ \frac{7}{2} mg R = \frac{1}{2} k x^2. \] (9)

Thus the initial displacement of the spring from equilibrium was

\[ x = \sqrt{\frac{7mg R}{k}}. \] (10)