A body of mass \( m \) is moving along the x-axis. Its potential energy is given by the function

\[
U(x) = b(x^2 - a^2)^2
\]

where \( b = 2 \text{J} \cdot \text{m}^{-4} \) and \( a = 1 \text{m} \).

**Finding the x-component of the Force from the Potential Energy:**

Recall the definition of potential energy

\[
U(x) - U(x_0) = -\int_{x_0}^{x} F_x(x') \, dx'
\]

The first fundamental theorem of calculus states that

\[
U(x) - U(x_0) = \int_{x_0}^{x} \frac{dU(x')}{dx'} \, dx'.
\]

Comparing Equation (2) with Equation (3) shows that the \( x \)-component of the force is the negative derivative (with respect to position) of the potential energy,
Question 1: What is the $x$-component of the force associated with the potential energy given by Equation (1)?

Answer:

To find the $x$-component of the force from the potential we use

$$F_x(x) = -\frac{dU(x)}{dx} = -4b(x^2 - a^2)x$$

Question 2 Make a sketch of the $x$-component of the force $F_x(x)$ vs. $x$.
**Extrema Points:**

The minimum and maximum points (the extrema) of the potential energy function occurs at the point where the first derivative vanishes

\[
\frac{dU(x)}{dx} = 0. \quad (5)
\]

**Question 3:** Find the minimum and maximum points of the potential energy.

**Answer**

The extrema points of the potential occur when

\[
\frac{dU(x)}{dx} = 0
\]

or equivalently when

\[
0 = F_x(x) = -2b(x^2 - a^2)2x.
\]

There are three extrema at \( x = 0 \), \( x = a \), and \( x = -a \).

**Stable Equilibrium Points:**

Suppose the potential energy function has positive curvature in the neighborhood of a extremum at \( x_0 \). If the body is moved a small distance to the point \( x > x_0 \) away from this point \( x_0 \), the slope of the potential energy function is positive, \( dU(x)/dx > 0 \); hence the component of the force is negative since \( F_x = -dU(x)/dx < 0 \). Thus the body experiences a restoring force towards the extremum point of the potential. If the body is moved a small distance to the point \( x < x_0 \) then \( dU(x)/dx < 0 \); the component of the force is positive since \( F_x = -dU(x)/dx > 0 \). Thus the body again experiences a restoring force towards the extremum point of the potential.

**Question 4** Which of these extrema points correspond to a stable equilibrium point?

**Answer:**

The points \( x = a \) and \( x = -a \) correspond to stable equilibrium points because if you move slightly away from these points the force is restoring, i.e. the direction of the force is always points towards the equilibrium point. The point \( x = 0 \) is an unstable equilibrium point. If you move slightly away from \( x = 0 \) in either direction, the force points away from \( x = 0 \). For
example consider values of $x > 0$, from the graph of $F_x(x)$ vs. $x$, the sign of the x-component of the force is positive hence the force points in the positive x-direction which is away from $x = 0$. If the object moves in the negative x-direction, then the x-component of the force is negative, which is again pointing away from $x = 0$. So $x = 0$ is an unstable equilibrium point. You can also use the potential energy graph to determine that $x = 0$ is an unstable equilibrium point.

towards the extrema of the potential energy. Such an extremum point is called a stable equilibrium point.

**Energy Diagram:**

The mechanical energy at any point $x$ is the sum of the kinetic energy $K(x)$ and the potential energy $U(x)$

$$E(x) = U(x) + K(x)$$ (6)

Both the kinetic energy and the potential energy are functions of the position of the body. Assume that the mechanical energy is a constant $E$, then for all points $x$

$$E(x) = E$$ (7)

The energy is a constant of the motion and can be either a positive value or zero. When the energy is zero, the body is at rest at the equilibrium positions, $x = a$ or $x = -a$.

A straight horizontal line on the plot of $U(x)$ vs. $x$ corresponds to a non-zero positive value for the energy $E$. The kinetic energy at a point $x_1$ is the difference between the energy and the potential energy,

$$K(x_1) = E - U(x_1)$$ (8)
At the points, where \( E = U(x) \), the kinetic energy is zero. Regions where the kinetic energy is negative, are called the \textit{classically forbidden regions}, which the body can never reach if subject to the laws of classical mechanics. In quantum mechanics, there is a very small probability that the body can be found in the classically forbidden regions.

\textbf{Question 5:} Describe what happens to the body if it has energy \( E > 2 \text{ J} \) and is located at \( x = 0 \).

\textbf{Answer:} At \( x = 0 \) the body has a positive kinetic energy hence is moving in either the positive or negative \( x \)-direction. As the object moves away from \( x = 0 \), the kinetic energy increases, whereas if the object moves slightly away from \( x = a \) or \( x = -a \), the kinetic energy decreases hence the speed decreases and the object slows down until it eventually stops where \( K = 0 \). However the \( x \)-component of the force is non-zero at those points (slope of \( U(x) \) vs \( x \) is non-zero), hence it reverses direction.

\textbf{Question 6:} Suppose the body starts with zero speed at \( x = 4a \). What is its speed at \( x = 0 \) and at \( x = -a \)?

\textbf{Answer:}

At \( x = 4a \), \( K(4a) = 0 \). Also

\[ U(4a) = b(16a^2 - a^2) = 225ba^4 = 225(2 \text{ J \cdot m}^{-4})(1\text{ m})^4 = 450 \text{ J}. \]

Therefore the value of the energy is

\[ E = E(4a) = K(4a) + U(4a) = 0 + 450 \text{ J} = 450 \text{ J}. \]
At $x = 0$

\[ U(0) = b(-a^2)^2 = ba^4 = (2 \cdot m^4)(a = 1m)^4 = 2J. \]

So

\[ K(0) = E - U(0) = 450 \cdot J - 2 \cdot J = 448 \cdot J. \]

Therefore at $x = 0$ the speed is

\[ v(0) = \sqrt{\frac{2K(0)}{m}} = \sqrt{\frac{896 \cdot J}{m}}. \]

Similarly at $x = -a$, $U(-a) = 0$. So

\[ K(-a) = E - U(-a) = 450 \cdot J - 0 = 450 \cdot J. \]

Therefore at $x = -a$, the speed is

\[ v(-a) = \sqrt{\frac{2K(-a)}{m}} = \sqrt{\frac{900 \cdot J}{m}}. \]