Jumping Off as Flatcar Solution

N people, each of mass $m_p$, stand on a railway flatcar of mass $m_c$. They jump off one end of the flatcar with velocity $u$ relative to the car. The car rolls in the opposite direction without friction.

a) What is the final velocity of the car if all the people jump at the same time?

b) What is the final velocity of the car if the people jump off one at a time?

c) Does case a) or b) yield the largest final velocity of the flat car.

Solution:

We begin by choosing a reference frame at rest with respect to the ground and identify our system as the flatcar and all the people. Since there are no external forces in the horizontal direction, the horizontal component of the momentum of the system is constant

\[ p_{x,i} = p_{x,f}. \] (1.1)

We can use this fact to solve for the final speed $v_f$ of the flatcar when all the people jump off together. We need to be careful to use the fact that the speed of each jumper relative to ground is given by $|u - v_f|$. We take as our initial state the car and people at rest. The final state is immediately after all the people have jumped off. The schematic momentum diagram below shows these states.

Then the initial x-component of the momentum is

\[ p_{x,i} = 0. \] (1.2)
The final x-component of the momentum is

\[ p_{x,f} = -m_c v_f + N m_p (u - v_f). \]  

(1.3)

Substituting Eq. (1.2) and Eq. (1.3) into Eq. (1.1) yields

\[ 0 = -m_c v_f + N m_p (u - v_f). \]  

(1.4)

We can solve Eq. (1.4) for the final velocity of the car,

\[ v_f = \frac{N m_p}{N m_p + m_c} u. \]  

(1.5)

(b) If the people jump off one at a time, we need to be more careful. Again the momentum of the system is constant but we have N jumps.

Before the first jump, the momentum is still zero. Immediately after the first person jumped, the x-component of the momentum is

\[ p_{x,f,1} = -((N - 1)m_p + m_c) v_{f,1} + m_p (u - v_{f,1}). \]  

(1.6)

Since the x-component of the momentum is constant we have that

\[ 0 = -((N - 1)m_p + m_c) v_{f,1} + m_p (u - v_{f,1}). \]  

(1.7)

We can solve this equation for the speed of the car after the first jump and find that

\[ v_{f,1} = \frac{m_p}{N m_p + m_c} u. \]  

(1.8)

Note that this is \(1/N\) of the speed found when the people all jumped at once (Eq. (1.5)).

Now let’s consider the second jump.
The x-component of the momentum before the jump is

\[ p_{x,j,2} = -((N-1)m_p + m_c) v_{f,1} \]  \hspace{1cm} (1.9)

The x-component of the momentum immediately after the second person jumped is

\[ p_{x,f,2} = -((N-2)m_p + m_c) v_{f,2} + m_p (u - v_{f,2}). \]  \hspace{1cm} (1.10)

Again applying the fact that the x-component of the momentum is constant yields

\[ -((N-1)m_p + m_c) v_{f,1} = -((N-2)m_p + m_c) v_{f,2} + m_p (u - v_{f,2}). \]  \hspace{1cm} (1.11)

We can rewrite this equation as

\[ -((N-1)m_p + m_c) v_{f,1} = -((N-1)m_p + m_c) v_{f,2} + m_p u. \]  \hspace{1cm} (1.12)

After dividing through by \(((N-1)m_p + m_c)\) and rearranging Eq. (1.12) becomes

\[ v_{f,2} = v_{f,1} + \frac{m_p}{(N-1)m_p + m_c} u. \]  \hspace{1cm} (1.13)

Substituting Eq. (1.8) into Eq. (1.13) yields the speed of the car immediately after the second person jumped off

\[ v_{f,2} = \frac{m_p}{Nm_p + m_c} u + \frac{m_p}{(N-1)m_p + m_c} u. \]  \hspace{1cm} (1.14)

Notice that the second term on the right hand side of Eq. (1.13) is larger than the first term on the right hand side, so the speed is now larger than \(v_{f,2} > 2v_{f,1} = \frac{2}{N} v_f\).
By induction, the speed after the \( j \)th person jumped off is

\[
v_{f,j} = \frac{m_p}{Nm_p + m_c}u + \frac{m_p}{(N-1)m_p + m_c}u + \cdots + \frac{m_p}{(N-(j-1))m_p + m_c}u. \tag{1.15}
\]

Hence the speed after the last person (the \( N \)th) person jumped off is

\[
v_{f,N} = \frac{m_p}{Nm_p + m_c}u + \frac{m_p}{(N-1)m_p + m_c}u + \cdots + \frac{m_p}{m_p + m_c}u. \tag{1.16}
\]

(c) To compare the answers to the previous two parts, note that each term in Eq.(1.16) is larger than the previous one, so we can conclude that

\[
v_{f,N} > v_f = \frac{Nm_p}{Nm_p + m_c}u. \tag{1.17}
\]

Without doing the calculation, we can alternatively use a proof by contradiction to understand why jumping one at a time produces a larger final velocity for the flatcar. Consider case A to be the everybody-jump-at-once case, and case B the one-at-a-time case. Let \( v_f \) and \( v_{f,N} \) be the final speed of the flatcar in cases A and B, respectively. Then, since each jumper is specified to have a speed \( u \) relative to the flatcar's speed immediately after his jump, in case A every jumper ends with an x-component of the velocity \( u - v_f \). Now suppose that \( v_f > v_{f,N} \). Then each jumper in case B has a final x-component of velocity greater than or equal to \( u - v_{f,N} \), and hence larger than the x-component of the jumpers in case A, which is \( u - v_f \). Thus the total x-component of the momentum of the jumpers in case B is greater than in case A, so the magnitude of x-component the recoil momentum of the flatcar must also be greater in case B (we need to take the magnitude of the x-component of the recoil momentum because the recoil is in the negative x-direction and so the x-component is negative). Thus we have contradicted our hypothesis. Similarly, if we suppose that \( v_f = v_{f,N} \), we could conclude that all the jumpers except the last in case B would have an x-component of momentum larger than each jumper in case A, so again we would have a contradiction. Thus, \( v_{f,N} > v_f \) is the only possibility.