IC_W08D2-5 Worked Example and Table Problem: Two and Three Ball Bounce Solutions

Consider two balls that are dropped from a height \( h_i \) above the ground, one on top of the other. Ball 1 is on top and has mass \( m_1 \), and ball 2 is underneath and has mass \( m_2 \) with \( m_2 > m_1 \). Assume that there is no loss of kinetic energy during all collisions. Ball 2 first collides with the ground and rebounds. Then, as ball 2 starts to move upward, it collides with ball 1 that is still moving downwards. How high will ball 1 rebound in the air?

Hint: consider this collision as seen by an observer moving upward with the same speed as ball 2 has after it collides with ground. What speed does ball 1 have in this reference frame after it collides with ball 2?

Solution:

The system consists of the two balls and the earth. There are five special states for this motion shown in the figure above.

Initial State: the balls are released from rest at a height \( h_i \) above the ground.

State A: the balls just reach the ground with speed \( v_a = \sqrt{2gh_i} \). This follows from \( \Delta E_{\text{mech}} = 0 \Rightarrow \Delta K = -\Delta U \). Thus \( (1/2)mv_a^2 - 0 = -mg\Delta h = mgh_i \Rightarrow v_a = \sqrt{2gh_i} \).
State B: immediately before the collision of the balls. Ball 2 has collided with the ground and
reversed direction with the same speed, \( v_a \), but ball 1 is still moving downward with speed \( v_a \).

State C: immediately after the collision of the balls. Because we are assuming that \( m_2 \gg m_1 \),
ball 2 does not change its speed as a result of the collision so it is still moving upward with speed \( v_a \). As a result of the collision, ball 1 moves upward with speed \( v_b \).

Final State: ball 1 reaches a maximum height \( h_f = \frac{v_b^2}{2g} \) above the ground. This again
follows from \( \Delta K = -\Delta U \Rightarrow 0 - (1/2)mv_b^2 = -mg\Delta h = -mgh_f \Rightarrow h_f = \frac{v_b^2}{2g} \).

Choice of Reference Frame:

As indicated in the hint above, this collision is best analyzed from the reference frame of an
observer moving upward with speed \( v_a \), the speed of ball 2 just after it rebounded with the
ground. In this frame immediately, before the collision, ball 1 is moving downward with a speed \( v'_b \) that is twice the speed seen by an observer at rest on the ground (lab reference frame).

\[ v'_b = 2v_a \]  \hspace{1cm} (1)

The mass of ball 2 is much larger than the mass of ball 1, \( m_2 \gg m_1 \). This enables us to consider
the collision (between States B and C) to be equivalent to ball 1 bouncing off a hard wall, while
ball 2 experiences virtually no recoil. Hence ball 2 remains at rest in the reference frame moving
upwards with speed \( v_a \) with respect to observer at rest on ground. Before the collision, ball 1 has
speed \( v'_b = 2v_a \). Since there is no loss of kinetic energy during the collision, the result of the
collision is that ball 1 changes direction but maintains the same speed,

\[ v'_b = 2v_a \]  \hspace{1cm} (2)

However, according to an observer at rest on the ground, after the collision ball 1 is moving
upwards with speed

\[ v_b = 2v_a + v_a = 3v_a. \]  \hspace{1cm} (3)

While rebounding, the mechanical energy of the smaller superball is constant (we consider the
smaller superball and the Earth as a system) hence between State C and the Final State,

\[ \Delta K + \Delta U = 0. \]  \hspace{1cm} (4)

The change in kinetic energy is

\[ \Delta K = -\frac{1}{2}m_1(3v_a)^2 \]  \hspace{1cm} (5)
The change in potential energy is
\[ \Delta U = m_1 g h_f. \] \hfill (6)

So the condition that mechanical energy is constant (Equation (4)) is now
\[-\frac{1}{2} m_1 (3v_a)^2 + m_1 g h_f = 0. \] \hfill (7)

We can rewrite Equation (7) as
\[ m_1 g h_f = \frac{1}{2} m_1 (v_a)^2. \] \hfill (8)

Recall that we can also use the fact that the mechanical energy doesn’t change between the Initial State and State A yielding an equation similar to Eq. (8),
\[ m_1 g h_i = \frac{1}{2} m_1 (v_a)^2. \] \hfill (9)

Now substitute the expression for the kinetic energy in Eq. (9) into Eq. (8) yielding
\[ m_1 g h_f = 9 m_1 g h_i. \] \hfill (10)

Thus ball 1 reaches a maximum height
\[ h_f = 9 h_i. \] \hfill (11)

**Three Superballs Colliding Solution**

When three superballs collides with \( m_3 >> m_2 >> m_1 \), we can use the same reasoning. If we assume that ball 2 moves upward with speed \( 3v_a \) after the collision with ball 3. Now switch to the reference frame moving upward with speed \( 3v_a \). In this frame ball 1 is moving towards ball 2 with speed \( 4v_a \). In this same reference frame, after the collision between ball 1 and ball 2, ball 1 is moving upward with speed \( 4v_a \). Now switch back to the ground frame in which ball 1 is moving upward with speed \( 7v_a \). Therefore ball 1 reaches a maximum height
\[ h_f = 49 h_i. \] \hfill (12)