Emptying a Freight Car Solution

A freight car of mass $m_c$ contains a mass of sand $m_s$. At $t = 0$ a constant horizontal force of magnitude $F$ is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = dm_s / dt$. Find the speed of the freight car when all the sand is gone. Assume that the freight car is at rest at $t = 0$.

Solution: Choose the positive $x$-direction to point in the direction that the car is moving. Let’s take as our system the amount of sand of mass $\Delta m_s$ that leaves the freight car during the time interval $[t, t+\Delta t]$, and the freight car and whatever sand is in it at time $t$.

At the beginning of the interval the car and sand are moving with speed $v$ so the $x$-component of the momentum at time $t$ is given by

$$p_x(t) = (\Delta m_s + m_c(t))v,$$  \hspace{1cm} (1)

where $m_c(t)$ is the mass of the car and sand in it at time $t$. Denote by $m_{c,0} = m_c + m_s$ where $m_c$ is the mass of the car and $m_s$ is the mass of the sand in the car at $t = 0$, and $m_s(t) = bt$ is the mass of the sand that has left the car at time $t$ since
$$m_s(t) = \int_0^t \frac{dm}{dt} dt = \int_0^t b dt = bt.$$  \hfill (2)

Thus

$$m_c(t) = m_{c,0} - bt = m_c + m_s - bt.$$  \hfill (3)

During the interval \([t, t + \Delta t]\), the small amount of sand of mass \(\Delta m_s\) leaves the car with the speed of the car at the end of the interval \(v + \Delta v\). So the \(x\)-component of the momentum at time \(t + \Delta t\) is given by

$$p_x(t + \Delta t) = (\Delta m_s + m_c(t))(v + \Delta v).$$  \hfill (4)

Throughout the interval a constant force \(F\) is applied to the car so

$$F = \lim_{\Delta t \to 0} \frac{p_x(t + \Delta t) - p_x(t)}{\Delta t}.$$  \hfill (5)

From our analysis above Eq. (5) becomes

$$F = \lim_{\Delta t \to 0} \frac{(m_c(t) + \Delta m_s)(v + \Delta v) - (m_c(t) + \Delta m_s)v}{\Delta t}.$$  \hfill (6)

Eq. (6) simplifies to

$$F = \lim_{\Delta t \to 0} m_c(t) \frac{\Delta v}{\Delta t} + \lim_{\Delta t \to 0} \frac{\Delta m_s \Delta v}{\Delta t}.$$  \hfill (7)

The second term vanishes when we take the \(\Delta t \to 0\) because it is of second order in the infinitesimal quantities (in this case \(\Delta m_s \Delta v\)) and so when dividing by \(\Delta t\) the quantity is of first order and hence vanishes since both \(\Delta m_s \to 0\) and \(\Delta v \to 0\). So Eq. (7) becomes

$$F = \lim_{\Delta t \to 0} m_c(t) \frac{\Delta v}{\Delta t}.$$  \hfill (8)

We now use the definition of the derivative:

$$\lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = \frac{dv}{dt}$$  \hfill (9)

in Eq. (8) which then becomes the differential equation

$$F = m_c(t) \frac{dv}{dt}.$$  \hfill (10)

Using Eq. (3) we have
\[ F = (m_c + m_s - bt) \frac{dv}{dt}. \]  \hfill (11)

(b) We can integrate this equation through the separation of variable technique. Rewrite Eq. (11) as

\[ dv = \frac{F dt}{(m_c + m_s - bt)}. \]  \hfill (12)

We can then integrate both sides of Eq. (12) with the limits as shown

\[ \int_{v'=0}^{v'=v(t)} dv' = \int_{t'=0}^{t'=t} \frac{F dt'}{m_c + m_s - bt'}. \]  \hfill (13)

Integration yields the velocity of the car as a function of time

\[ v(t) = - \frac{F}{b} \ln(m_c + m_s - bt') \bigg|_{v'=t' = 0}^{v'=t} = - \frac{F}{b} \ln \left( \frac{m_c + m_s - bt}{m_c + m_s} \right) \]