Two point-like particles of equal mass are rotating at a constant angular speed about point A about its center as shown. How does the angular momentum about the point B compare to the angular momentum about the point A? What about at a later time when the particles have rotated by 90 degrees?

**Solution.** The total momentum of the system consisting of the two particles is zero. Therefore the angular momentum is independent of the point that it is calculated about.

Choose a Cartesian coordinate system

The angular momentum about point A is

\[ \mathbf{L}_A = \mathbf{r}_{A,1} \times m_1 \mathbf{v}_1 + \mathbf{r}_{A,2} \times m_2 \mathbf{v}_2 = r(-\hat{i}) \times mv(-\hat{j}) + r(+\hat{i}) \times mv(+\hat{j}) = 2mr\hat{k} \]

(1)

The angular momentum about point B is

\[ \mathbf{L}_B = \mathbf{r}_{B,1} \times m_1 \mathbf{v}_1 + \mathbf{r}_{B,2} \times m_2 \mathbf{v}_2 = (2r+d)(-\hat{i}) \times mv(-\hat{j}) + d(-\hat{i}) \times mv(+\hat{j}) = (mv(2r+d) - mv\hat{k}) = 2mr\hat{k} \]

(2)

More generally:
Consider a system of N particles, and two points $A$ and $B$.

The angular momentum of the ith particle about the point $A$ is given by

$$
\vec{L}_{A,i} = \vec{r}_{A,i} \times m_i \vec{v}_i.
$$  \hspace{1cm} (3)

So the angular momentum of the system of particles about the point $A$ is given by the sum

$$
\vec{L}_A = \sum_{i=1}^{N} \vec{L}_{A,i} = \sum_{i=1}^{N} \vec{r}_{A,i} \times m_i \vec{v}_i.
$$  \hspace{1cm} (4)

The angular momentum about the point $B$ can be calculated in a similar way and is given by

$$
\vec{L}_B = \sum_{i=1}^{N} \vec{L}_{B,i} = \sum_{i=1}^{N} \vec{r}_{B,i} \times m_i \vec{v}_i.
$$  \hspace{1cm} (5)

From the above figure, the vectors

$$
\vec{r}_{A,B} = \vec{r}_{A,i} + \vec{r}_{A,B}.
$$  \hspace{1cm} (6)

We can substitute Eq. (6) into Eq. (4) yielding

$$
\vec{L}_A = \sum_{i=1}^{N} (\vec{r}_{B,i} + \vec{r}_{A,B}) \times m_i \vec{v}_i = \sum_{i=1}^{N} \vec{r}_{B,i} \times m_i \vec{v}_i + \sum_{i=1}^{N} \vec{r}_{A,B} \times m_i \vec{v}_i.
$$  \hspace{1cm} (7)

The first term in Eq. (7) is the angular momentum about the point $B$. The vector $\vec{r}_{A,B}$ is a constant and so can be pulled out of the sum in the second term, and Eq. (7) becomes
The sum in the second term is the momentum of the system

\[ \mathbf{p}_{\text{sys}} = \sum_{i=1}^{N} m_i \mathbf{v}_i. \]  

Therefore the angular momentum about the points \( A \) and \( B \) are related by

\[ \mathbf{L}_A = \mathbf{L}_B + \mathbf{r}_{A,B} \times \mathbf{p}_{\text{sys}}. \]  

Thus if the momentum of the system is zero, the angular momentum is the same about any point. In particular, the momentum of a system of particles is zero by definition in the center of mass reference frame. Hence the angular momentum is the same about any point in the center of mass reference frame.

When the particles have rotated 90 degrees, the momentum of the system is still zero so the angular momentum is independent of the point.