A long narrow uniform stick of length $l$ and mass $m$ lies motionless on ice (assume the ice provides a frictionless surface). The center of mass of the stick is the same as the geometric center (at the midpoint of the stick). The moment of inertia of the stick about its center of mass is $I_{cm}$. A puck (with putty on one side) has the same mass $m$ as the stick. The puck slides without spinning on the ice with a speed of $v_0$ toward the stick, hits one end of the stick, and attaches to it. You may assume that the radius of the puck is much less than the length of the stick so that the moment of inertia of the puck about its center of mass is negligible compared to $I_{cm}$.

**Solution:**

In this problem we will calculate the center of mass of the puck-stick system after the collision. There are no external forces or torques acting on this system so the momentum of the center of mass is constant before and after the collision and the angular momentum about the center of mass of the puck-stick system is constant before and after the collision.
collision. We shall use these relations to compute the final angular velocity of the puck-stick about the center of mass. We note that the mechanical energy is not constant because the puck collides completely inelastically with the stick.

a) With respect to the center of the stick, the center of mass of the stick-puck combination is (neglecting the radius of the puck)

\[ d_{cm} = \frac{m_{\text{stick}}d_{\text{stick}} + m_{\text{puck}}d_{\text{puck}}}{m_{\text{stick}} + m_{\text{puck}}} = \frac{m(0) + m(l/2)}{m + m} = \frac{l}{4}. \]  

b) During the collision, the only net forces on the system (the stick-puck combination) are the internal forces between the stick and the puck (transmitted through the putty). Hence, linear momentum is conserved. Initially only the puck had linear momentum \( p_0 = mv_0 \). After the collision, the center of mass of the system is moving with speed \( v_f \). Equating initial and final linear momenta,

\[ mv_0 = (2m)v_f \quad \Rightarrow \quad v_f = \frac{v_0}{2}. \]  

The direction of the velocity is the same as the initial direction of the puck’s velocity.

Note that the result of part a) was not needed for part b); if the masses are the same, Equation (0.2) would hold for any mass distribution of the stick.

c) The forces that deform the putty do negative work (the putty is compressed somewhat), and so mechanical energy is not conserved; the collision is totally inelastic.

d) Choose the center of mass of the stick-puck combination, as found in part a), as the point about which to find angular momentum. This choice means that after the collision there is no angular momentum due to the translation of the center of mass. Before the collision, the angular momentum was entirely due to the motion of the puck,
\[
\mathbf{\vec{L}}_0 = \mathbf{\vec{r}}_{\text{puck}} \times \mathbf{\vec{p}}_0 = (l/4)\left(m v_0\right)\mathbf{\hat{k}},
\]

where \(\mathbf{\hat{k}}\) is directed out of the page in the figure above. After the collision, the angular momentum is

\[
\mathbf{\vec{L}}_f = I_{\text{cm}} \omega_f \mathbf{\hat{k}},
\]

where \(I_{\text{cm}}\) is the moment of inertia about the center of mass of the stick-puck combination. This moment of inertia of the stick about the new center of mass is found from the parallel axis theorem, and the moment of inertia of the puck is \(m(l/4)^2\), and so

\[
I_{\text{cm}} = I_{\text{cm}},\text{stick} + I_{\text{cm}},\text{puck} = \left(I_{\text{cm}} + m(l/4)^2\right) + m(l/4)^2 = I_{\text{cm}} + \frac{ml^2}{8}.
\]

Inserting this expression into Equation (0.4), equating the expressions for \(\mathbf{\vec{L}}_0\) and \(\mathbf{\vec{L}}_f\) and solving for \(\omega_f\) yields

\[
\omega_f = \frac{m(l/4)}{I_{\text{cm}} + ml^2/8} v_0.
\]

If the stick is uniform, \(I_{\text{cm}} = ml^2/12\) and Equation (0.6) reduces to

\[
\omega_f = \frac{6}{5} \frac{v_0}{l}.
\]

It may be tempting to try to calculate angular momentum about the contact point, where the putty hits the stick. If this is done, there is no initial angular momentum, and after the collision the angular momentum will be the sum of two parts, the angular momentum of the center of mass of the stick and the angular momentum about the center of the stick,

\[
\mathbf{\vec{L}}_f = \mathbf{\vec{r}}_{\text{cm}} \times \mathbf{\vec{p}}_{\text{cm}} + I_{\text{cm}} \mathbf{\omega}_f.
\]

There are two crucial things to note: First, the speed of the center of mass is not the speed found in part b); the rotation must be included, so that \(v_{\text{cm}} = v_0 / 2 - \omega_f (l/4)\).

Second, the direction of \(\mathbf{\vec{r}}_{\text{cm}} \times \mathbf{\vec{p}}_{\text{cm}}\) with respect to the contact point is, from the right-hand rule, \(\text{into}\) the page, or the \(-\mathbf{\hat{k}}\) -direction, opposite the direction of \(\mathbf{\omega}_f\). This is to be expected, as the sum in Equation (0.8) must be zero. Adding the \(\mathbf{\hat{k}}\) -components (the only components) in Equation (0.8),
\[-(l/2)m\left(v_0/2 - \omega_f(l/4)\right) + I_{cm}\omega_f = 0. \quad (0.9)\]

Solving Equation (0.9) for \( \omega_f \) yields Equation (0.6).

This alternative derivation should serve two purposes. One is that it doesn’t matter which point we use to find angular momentum. The second is that use of foresight, in this case choosing the center of mass of the system so that the final velocity does not contribute to the angular momentum, can prevent extra calculation. It’s often a matter of trial and error (“learning by misadventure”) to find the “best” way to solve a problem.

e) The time of one rotation will be the same for all observers, independent of choice of origin. This fact is crucial in solving problems, in that the angular velocity will be the same (this was used in the alternate derivation for part d) above). The time for one rotation is the period \( T = 2\pi / \omega_f \) and the distance the center of mass moves is

\[
x_{cm} = v_{cm} T = 2\pi \frac{v_{cm}}{\omega_f} = 2\pi \frac{v_0/2}{I_{cm} + m(l/4)^2} v_0 = 2\pi \frac{I_{cm} + m l^2 / 8}{m(l/2)}.
\quad (0.10)
\]

Using \( I_{cm} = ml^2 / 12 \) for a uniform stick gives

\[
x_{cm} = \frac{5}{6} \pi l.
\quad (0.11)
\]