IC-W12D2-5 Table Problem Oscillating Pivoted Ring Solution

A physical pendulum consists of a ring of radius $R$ and mass $m$. The ring is pivoted (assume no energy is lost in the pivot). The ring is pulled out such that its center of mass makes an angle $\theta_0$ from the vertical and released from rest. The gravitational constant is $g$.

a) First assume that $\theta_0 \ll 1$. Show that the angle the center of mass makes with the vertical satisfies a simple harmonic oscillator differential equation. What is the angular frequency of oscillation?

b) What is the angular speed of the ring at the bottom of its swing?

Solution: (a) We shall calculate torque about the pivot point (the pivot forces therefore do not contribute to the torque). A torque diagram is shown in the figure below.

Using the torque method we have that

$$-MgR\sin\theta = I_p \frac{d^2\theta}{dt^2}. \quad (1)$$

We use the parallel axis theorem to calculate the moment of inertia about the pivot point,

$$I_p = I_{cm} + MR^2 = MR^2 + MR^2 = 2MR^2. \quad (2)$$
Therefore the equation of motion (Eq. (1)) becomes

\[ \frac{d^2\theta}{dt^2} + \frac{g}{2R} \sin \theta = 0. \quad (3) \]

Using the small angle approximation that \( \sin \theta = \theta \), we have that the pivoted ring undergoes approximate simple harmonic motion

\[ \frac{d^2\theta}{dt^2} + \frac{g}{2R} \theta = 0, \quad (4) \]

where the angular frequency of small oscillations is given by

\[ \omega_0 = \sqrt{\frac{g}{2R}}. \quad (5) \]

b) The general solution for the angle \( \theta(t) \) and the component of the initial angular velocity are given by

\[ \theta(t) = \theta_0 \cos(\omega_0 t) + \frac{\dot{\theta}_0}{\omega_0} \sin(\omega_0 t) \quad (6) \]

\[ \frac{d\theta}{dt}(t) = -\omega_0 \theta_0 \sin(\omega_0 t) + \dot{\theta}_0 \cos(\omega_0 t). \quad (7) \]

where we denote the component of the initial angular velocity by \( \dot{\theta}_0 = \frac{d\theta}{dt}(t = 0) \). From the description of the problem \( \dot{\theta}_0 = \frac{d\theta}{dt}(t = 0) = 0 \). Therefore the general solutions simply to

\[ \theta(t) = \theta_0 \cos(\omega_0 t) \quad (8) \]

\[ \frac{d\theta}{dt}(t) = -\omega_0 \theta_0 \sin(\omega_0 t). \quad (9) \]

The ring is at the bottom of its swing when \( \theta(t_f) = \theta_0 \cos(\omega_0 t_f) = 0 \). This first occurs when \( \omega_0 t_f = \pi / 2 \). At that instant the component of the angular speed is

\[ \frac{d\theta}{dt}(t_f) = -\omega_0 \theta_0 = -\sqrt{\frac{g}{2R}} \theta_0. \quad (10) \]