IC-W13D1-4 Rolling Without Slipping Oscillating Cylinder Solution

Attach a solid cylinder of mass $M$ and radius $R$ to a horizontal massless spring with spring constant $k$ so that it can roll without slipping along a horizontal surface. At time $t$, the center of mass of the cylinder is moving with speed $V_{cm}$ and the spring is compressed a distance $x$ from its equilibrium length. What is the period of simple harmonic motion for the center of mass of the cylinder?

Solution: At time $t$, the energy of the rolling cylinder and spring system is

$$E = \frac{1}{2} M V_{cm}^2 + \frac{1}{2} I_{cm} \left( \frac{d\theta}{dt} \right)^2 + \frac{1}{2} k x^2$$

where $x$ is the amount the spring has compressed, $I_{cm} = \frac{1}{2} MR^2$, and because it is rolling without slipping

$$\frac{d\theta}{dt} = \frac{V_{cm}}{R}.$$ 

Therefore the energy is

$$E = \frac{1}{2} MV_{cm}^2 + \frac{1}{4} MR^2 \left( \frac{V_{cm}}{R} \right)^2 + \frac{1}{2} k x^2 = \frac{3}{4} MV_{cm}^2 + \frac{1}{2} k x^2.$$ 

The energy is constant so

$$0 = \frac{dE}{dt} = \frac{3}{4} 2MV_{cm} \frac{dV_{cm}}{dt} + \frac{1}{2} k 2x \frac{dx}{dt} = V_{cm} \left( \frac{3}{2} M \frac{d^2x}{dt^2} + kx \right)$$

Because $V_{cm}$ is non-zero most of the time, the displacement of the spring satisfies a simple harmonic oscillator equation

$$\frac{d^2x}{dt^2} + \frac{2k}{3M} x = 0.$$ 

Hence the period is

$$T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{3M}{2k}}.$$