W14D1-4 Rolling Wheel

A thin hoop of mass $m$ and radius $R$ rolls without slipping about the $z$ axis. It is supported by an axle of length $b$ through its center. The hoop circles around the $z$ axis with angular speed $\Omega$. (Note: the moment of inertia of a hoop for an axis along its diameter is $(1/2)mR^2$.) What is the normal force of the ground on the hoop? What is the vertical force on the axle at the joint where the axle meets the $z$-axis?
\[ v_{cm} = b \omega \] center of mass circular motion

\[ v_{cm} = R \omega \] rolling without slipping

\[ \Rightarrow \quad F \omega = b \omega \]

\[ \vec{L}_S = \vec{r}_{cm} \times m \vec{v} + \vec{l}_{cm} = bm v_{cm} \hat{k} + -I_1 \omega \hat{r} + I_2 \omega \hat{k} \]

\[ \vec{\tau}_S = \frac{d \vec{L}_S}{dt} \]

\[ (bm g - b \vec{n}) \hat{\theta} = -I_1 \omega \vec{r} \hat{\theta} \]

\[ bm g - b \vec{n} = -I_1 \omega \vec{r} \]

\[ F \omega = b \omega \]

\[ \Rightarrow \quad N = mg + I_1 \frac{\omega^2}{\ell} \]

\[ F = m \ddot{\ell} \]

\[ \vec{k} : -F_V + N - mg = 0 \]

\[ F_V = m g + N = m g + (mg + I_1 \frac{\omega^2}{\ell}) \]

\[ F_V = I_1 \frac{\omega^2}{\ell} \]
\[ \overline{\tau}_{cm} = \frac{d\overline{I}_{cm}}{dt} \Rightarrow b \ F_v(-\dot{\theta}) = I_1 \omega_{\overline{I}}(-\dot{\theta}) \]

\[ b \ F_v(-\dot{\theta}) = \frac{I_1 \ b \ \omega^2}{\bar{r}}(-\dot{\theta}) \]

\[ \Rightarrow \ F_v = \frac{I_1 \ \overline{\omega}^2}{\bar{r}} \]