When an automobile rounds a curve at high speed (in the figure below the car is turning left), the loading (weight distribution) on the wheels is markedly changed. For sufficiently high speeds the loading on the inside wheel goes to zero, at which point the car starts to roll over. The tendency can be avoided by mounting a large spinning flywheel on the car.

a) What should be the sense of rotation of the flywheel to help equalize the loading? (Be sure that your method works for cars turning in either direction.)

b) Show that for a disk-shaped flywheel of mass $m_w$ and radius $R$, the requirement for equal loading is that the angular velocity of the flywheel, $\omega$, is related to the speed of the car $v_{cm}$ by

$$\omega = 2v_{cm} \frac{m_w h}{m_c R^2},$$

where $m_c$ is the total mass of the car and flywheel, and $h$ is the height of the center of mass of the car (including the flywheel) above the road. Assume the road is unbanked.
**Solution:** When the car is turning left, the free body diagram on the car is shown in the figure below.

![Free body diagram of the car](image)

The torque about the center of mass is then

\[
\vec{\tau}_{cm} = \vec{r}_{cm,1} \times (\vec{N}_1 + \vec{f}_1) + \vec{r}_{cm,2} \times (\vec{N}_2 + \vec{f}_2). \tag{1}
\]

The torques about the center of mass due to the forces \(\vec{N}_1, \vec{f}_1,\) and \(\vec{f}_2\) are all pointing into the page in the figures below.

![Torque diagram](image)

The torque about the center of mass due to \(\vec{N}_2\) is out of the page of the figure. This creates an unequal balance of loads on the tires, the load on the tires on the right side is greater than the load on the tires on the left side. It’s possible to balance this load by introducing an additional torque in the same direction as torque about the center of mass due to \(\vec{N}_2\) out of the page of the figure. This can be accomplished by mounting a flywheel in the following way.

Suppose the flywheel is mounted as shown in the figure below, with the flywheel spinning so that the angular momentum about the center of mass points to the right (the car is moving into the page in the figure).
Suppose the car is turning left and that we approximate the turn as a circle of radius $r$. Then the direction of the change in the spin angular momentum of the flywheel is in the direction of motion.

In order to effect this change, the car must exert a torque on the flywheel in this direction (into the page).

The flywheel exerts a counter torque on the car that is opposite in direction hence points opposite this direction (out of the page).

This counter torque increases the load on the wheels on the left side of the car, equalizing the imbalance in the load due to turning left.
Mounting the flywheel in this manner also stabilizes the load on the wheels for a turn to the right. Then the spin angular momentum of the flywheel changes in a direction opposite the motion of the car.

The car then exerts a torque on the flywheel pointing in the same direction as the direction of the change in spin angular momentum (out of the page in the figure below).

Thus the flywheel exerts a counter torque on the car in the direction of motion that equalizes the imbalance in the load on the wheels.

The only component of the angular momentum of the car about the center of mass that is changing when the car turns left is the spin angular momentum of the flywheel.
At the instant shown in the overhead figure, (with choice of unit vectors as shown) this contribution to the angular momentum about the center of mass is pointing in the positive $\hat{i}$-direction.

\[ \vec{L}_{cm,s} = I_s \omega_s \hat{i}. \]  

(2)

As the car turns left, the change in the spin angular momentum is in the positive $\hat{j}$-direction.

\[ \frac{d\vec{L}_{cm,s}}{dt} = I_s \omega_s \Omega \hat{j}. \]  

(3)

Thus the car must exert a torque on the flywheel equal to

\[ \vec{\tau}_{c,f} = \frac{d\vec{L}_{cm,s}}{dt} = I_s \omega_s \Omega \hat{j}. \]  

(4)

The flywheel exerts a counter torque on the car equal to

\[ \vec{\tau}_{f,c} = -\vec{\tau}_{c,f} = -I_s \omega_s \Omega \hat{j}. \]  

(5)

If we assume that the moment inertia of the flywheel is $I_s = (1/2)m_sR^2$, and that the angular speed about the circle is related to the center of mass speed by $\Omega = \frac{v_{cm}}{r}$, Eq. (5) becomes

\[ \vec{\tau}_{f,c} = -\frac{m_s R^2 \omega_s v_{cm}}{2r} \hat{j}. \]  

(6)

The forces along with our choice of unit vectors are shown in the figure below.
Because the car is not stable (not rotating about the $\hat{j}$-direction), the torque about the center of mass on the car is zero and is given by

$$\vec{\tau} = \vec{r}_{cm} = \vec{r}_{cm,1} \times (\vec{N}_1 + \vec{f}_1) + \vec{r}_{cm,2} \times (\vec{N}_2 + \vec{f}_2) + \vec{r}_{f,c}$$

$$= \frac{d}{2} (\vec{N}_1 - \vec{N}_2) \hat{j} + h(\vec{f}_1 + \vec{f}_2) \hat{i} - \frac{m_w R^2 \omega \vec{v}_{cm}}{2r} \hat{j} \cdot$$

When the loads are balanced $N_2 = N_1$ and so Eq. (7) becomes

$$\vec{\tau} = h(\vec{f}_1 + \vec{f}_2) \hat{j} - \frac{m_w R^2 \omega \vec{v}_{cm}}{2r} \hat{j} \cdot$$

The center of mass is accelerating toward the center of the circle, Newton’s Second Law applied in the to the center of mass of the car yields for the inward direction (negative $\hat{i}$-direction in the figure)

$$\vec{f}_1 + \vec{f}_2 = m_v \vec{v}^2_{cm} / r \cdot$$

We can substitute Eq. (9) into Eq. (8) and we find that

$$\frac{hm_v \vec{v}^2_{cm}}{r} = \frac{m_w R^2 \omega \vec{v}_{cm}}{2r} \cdot$$

We can solve Eq. (10) for the angular speed of the flywheel so that it has equal load on the wheels when the car is traveling at a speed $v_{cm}$

$$\omega_s = \frac{2hm_v \vec{v}_{cm}}{m_w R^2} \cdot$$