In a mill, grain is ground by a massive wheel that rolls without slipping in a circle on a flat horizontal mill stone driven by a vertical shaft. The rolling wheel has mass $M$, radius $b$ and is constrained to roll in a horizontal circle of radius $R$ at angular speed $\Omega$. The wheel pushes down on the lower mill stone with a force equal to twice its weight (normal force). The mass of the axle of the wheel can be neglected. Express your answers to the following questions in terms of $R$, $b$, $M$, $\Omega$, and $g$ as needed. The goal of this problem is to find $\Omega$.

a) What is the relation between the angular speed $\omega$ of the wheel about its axle and the angular speed $\Omega$ about the vertical axis?

b) Find the time derivative of the angular momentum about the joint (about the point $P$ in the figure above) $dL_P/dt$.

c) What is the torque about the joint (about the point $P$ in the figure above)?

d) What is the value of $\Omega$?

Solution: The figure below shows the pivot point along with some convenient coordinate axes.
\[ v_{cm} = b\omega. \quad (1) \]

Also the speed of the center of mass is related to the angular speed about the vertical axis associated with the circular motion of the center of mass by

\[ v_{cm} = R\Omega. \quad (2) \]

Therefore equating Eqs. (1) and (2) we have that

\[ \omega = \frac{\Omega R}{b}. \quad (3) \]

b) Assuming a uniform millwheel, \( I_{cm} = (1/2)Mb^2 \), the magnitude of the horizontal component of the angular momentum about the center of mass is

\[ L_{cm,h} = I_{cm}\omega = \frac{1}{2}Mb^2\omega = \frac{1}{2}\Omega MMRb. \quad (4) \]

The horizontal component of \( \vec{L}_{cm} \) is directed inward, and in vector form \( \vec{L}_{cm} = L_{cm,h}(-\hat{r}) \) in the above coordinate system.

c) The axle exerts both a force and torque on the wheel, and this force and torque would be quite complicated. That’s why we consider the forces and torques on the axle/wheel combination. The normal force of the wheel on the ground is equal in magnitude to \( N_{WG} = 2mg \) so the third-law counterpart, the normal force of the ground on the wheel has the same magnitude \( N_{GW} = 2mg \).

The joint (or hinge) at point \( P \) therefore must exert a force \( \vec{F}_{HA} \) on the end of the axle that has two components forces an inward force \( \vec{F}_2 \) to maintain the circular motion and a downward force \( \vec{F}_1 \) to reflect that the upward normal force is larger in magnitude than the weight.

\[ \]

\[ N_{GW} = 2mg \]

d) About point \( P \), \( \vec{F}_{HA} \) exerts no torque. The normal force exerts a torque of magnitude \( N_{GW}R = 2mgR \), directed out of the page, or, in vector form, \( \vec{\tau}_{p,N} = -2mg\hat{r} \). The weight exerts a toque of magnitude \( MgR \), directed into the page, or, in vector form, \( \vec{\tau}_{p,mg} = Mg\hat{r} \). So the torque about \( P \) is
\[
\tau_p = \tau_{p,N} + \tau_{p,mg} = -2mgR\dot{\theta} + MgR\ddot{\theta} = -MgR\ddot{\theta}.
\] (5)

As the wheel rolls, the horizontal component of the angular momentum about the center of mass will rotate, and the inward-directed vector will change in the negative \( \hat{\theta} \)-direction. Mathematically,

\[
\frac{d\tilde{L}_{cm,h}}{dt} = \left| \tilde{L}_{cm,h} \right| \Omega(-\dot{\theta}) = \frac{1}{2} \Omega MRb\dot{\Omega}(-\dot{\theta}),
\] (6)

where we used Eq. (4) for the magnitude of the horizontal component of the angular momentum about the center of mass. This is consistent with the torque about \( P \) pointing out of the page in the above figure. We can now apply the torque condition that

\[
\tau_p = \frac{d\tilde{L}_p}{dt}
\] (7)

that becomes using Eqs. (5) and (6)

\[
MgR(-\dot{\theta}) = \frac{1}{2} \Omega^2 MRb(-\dot{\theta})
\] (8)

We can now solve Eq. (8) for the angular speed about the vertical axis

\[
\Omega = \sqrt{\frac{2g}{b}}.
\] (9)