A rigid body is composed of a uniform disk (mass $m$, radius $R$) and a uniform rod (mass $m$, length $D$) that is rigidly fixed to the center of the disk. This body is pivoted about the center of the disk around a horizontal axis that is perpendicular to the plane of the page. Assume the pivot is frictionless and the acceleration due to gravity is $g$.

a) Find the moment of inertia $I_{\text{pivot}}$ about the pivot point.

b) Suppose the pendulum is swinging freely back and forth. Write down an expression for the angular acceleration about the pivot point. You may leave your answer in terms of $m$, $g$, $R$, $I_{\text{pivot}}$, $D$ and the angle $\theta$ as needed.

c) Suppose the angle $\theta$ is small throughout the motion. That is, you may assume $\sin \theta = \theta$ and $\cos \theta = 1$. What is the period for this pendulum? Express your answer in terms of $m$, $R$, $D$, $g$ and $I_{\text{pivot}}$.

d) Now suppose there is no restriction on the value of $\theta$ (it can be large). What is the minimum angular speed $\omega_{\text{min}}$ that the pendulum should have at the bottom of its swing so that the pendulum can revolve completely around the pivot point?

Solution:

a) From the parallel axis theorem, or a handy formula sheet, the moment of inertia of the rod about the pivot point is $I_{\text{rod,pivot}} = mD^2 / 3$. The pivot is the center of the disc, so $I_{\text{disc,pivot}} = mR^2 / 2$ and the total moment of inertial about the pivot is $I_{\text{pivot}} = m(R^2 / 2 + D^2 / 3)$. 
b) The weight of the disc (and any contact force between the disc bearings and the pendulum) exert no torque, and the torque exerted by the weight of the rod, directed into the page in the figure, is \( \tau = mg(D/2) \sin \theta \). The angular acceleration is then

\[
\alpha = -\frac{\tau}{I_{\text{pivot}}} = -\frac{mg(D/2) \sin \theta}{m(R^2/2 + D^2/3)} = -\frac{g \sin \theta}{R^2/D + 2D/3},
\]

with the negative signs indicating a restoring torque.

c) The square of the frequency of small oscillations is given by the negative of the term multiplying \( \sin \theta \) in part (b), and so the period of small oscillations is

\[
T = 2\pi \sqrt{\frac{R^2/D + 2D/3}{g}}.
\]

d) Without the small angle approximation, this part of the problem cannot be solved directly by using torques; energy considerations must be used. At the bottom of the swing, the kinetic energy is \((1/2)I_{\text{pivot}} \omega_{\text{min}}^2\) and to just make it around the pivot point, the kinetic energy at the top should be taken to be zero. Note that the center of mass of the disc does not move, so in going from the bottom to the top, the change in gravitational potential energy is due only to the change in height of the center of mass of the rod and hence increased by \( \Delta U = mgD \). Therefore setting the change in kinetic energy equal to the negative of the change in potential energy,

\[
\frac{1}{2} I_{\text{pivot}} \omega_{\text{min}}^2 = mgD
\]

\[
\omega_{\text{min}}^2 = \frac{mgD}{I_{\text{pivot}}} = \frac{2gD}{R^2/2 + D^2/3}.
\]

Therefore the angular speed is

\[
\omega_{\text{min}} = \sqrt{\frac{2gD}{R^2/2 + D^2/3}}.
\]