# Circular Motion Kinematics

8.01

W04D1

Today's Reading Assignment:
MIT 8.01 Course Notes

Chapter 6 Circular Motion

Sections 6.1-6.2

# Announcements

Math Review Week 4 Tuesday 9-11 pm in 26-152.

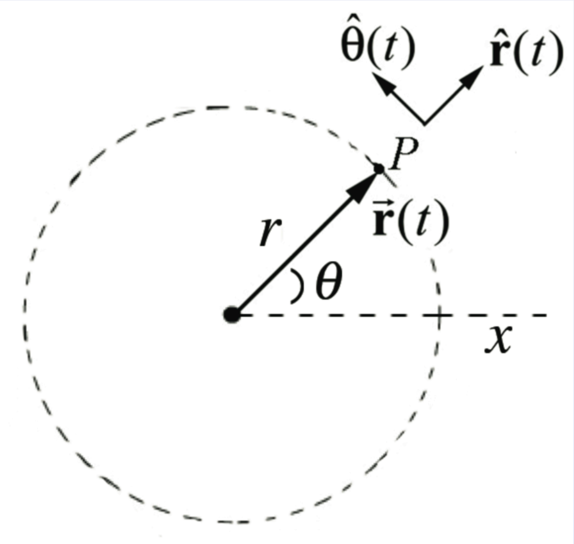
Next Reading Assignment (W04D2):

MIT 8.01 Course Notes
Chapter 9 Circular Motion Dynamics
Sections 9.1-9.2

# Kinematics in Two-Dimensions:
# Circular Motion

# Polar Coordinate System

Coordinates $(r,\theta)$




Unit vectors $(\hat{\mathbf{r}},\hat{\boldsymbol{\theta}})$

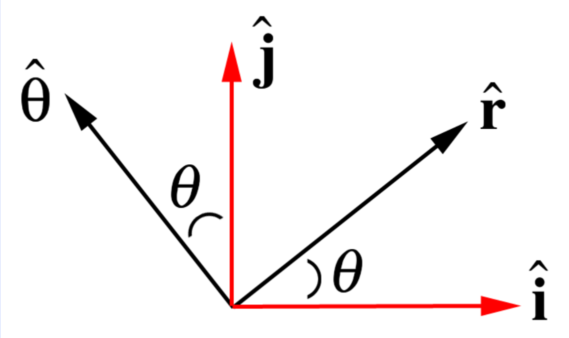
Relation to Cartesian Coordinates

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y/x)$$

$$\hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$$

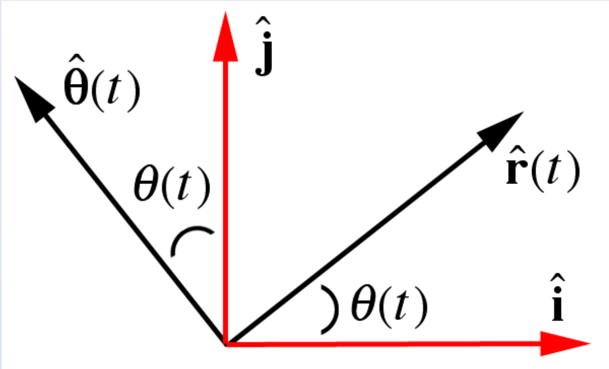


# Coordinate Transformations

Transformations between unit vectors in polar coordinates and Cartesian unit vectors

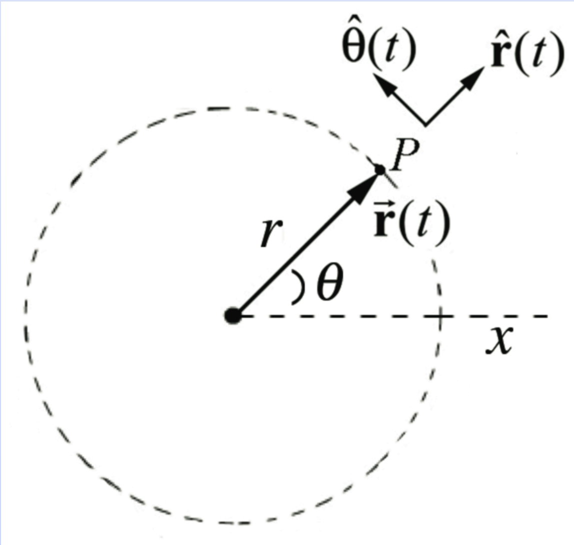
$$\hat{\mathbf{r}}(t) = \cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}$$

$$\hat{\boldsymbol{\theta}}(t) = -\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}$$




$$\hat{\mathbf{i}} = \cos\theta(t)\hat{\mathbf{r}}(t) - \sin\theta(t)\hat{\boldsymbol{\theta}}(t)$$

$$\hat{\mathbf{j}} = \sin\theta(t)\hat{\mathbf{r}}(t) + \cos\theta(t)\hat{\boldsymbol{\theta}}(t)$$



# Concept Question: Time Derivative of Position Vector for Circular Motion

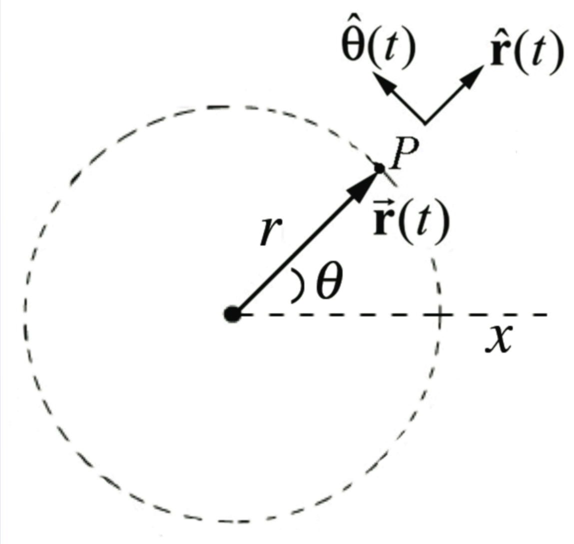
A point-like object undergoes circular motion at a constant speed. The vector from the center of the circle to the object

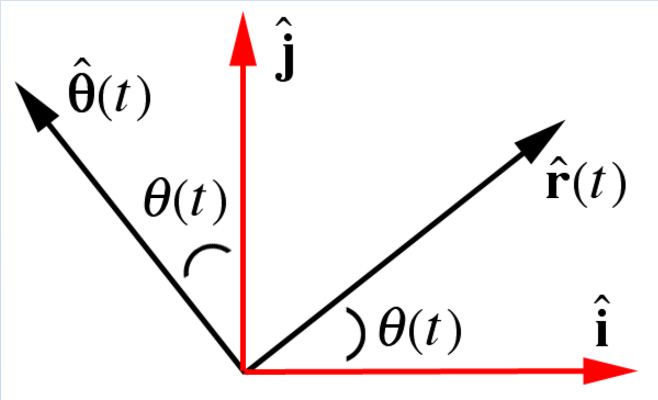
1. has constant magnitude and hence is constant in time.
2. has constant magnitude but is changing direction so is not constant in time.
3. is changing in magnitude and hence is not constant in time.

# Circular Motion: Position

Position

$$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t) = r(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}})$$



# Chain Rule of Differentiation

Recall that when taking derivatives of a differentiable function $f = f(\theta)$ whose argument is also a differentiable function $\theta = g(t)$ then $f = f(g(t)) = h(t)$ is a differentiable function of *t* and

$$\frac{df}{dt} = \frac{df}{d\theta}\frac{d\theta}{dt}$$

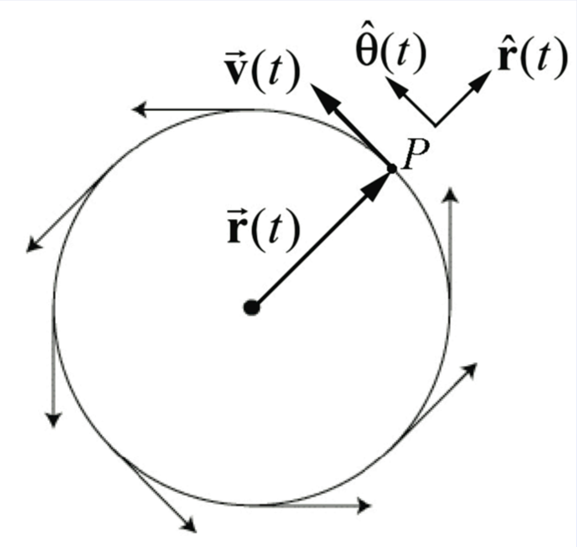
Examples:

$$\frac{d}{dt}\cos\theta(t) = -\sin\theta\frac{d\theta}{dt}$$

$$\frac{d}{dt}\sin\theta(t) = \cos\theta\frac{d\theta}{dt}$$

# Circular Motion: Velocity

Velocity $\vec{\mathbf{v}}(t) = r\dfrac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t)$

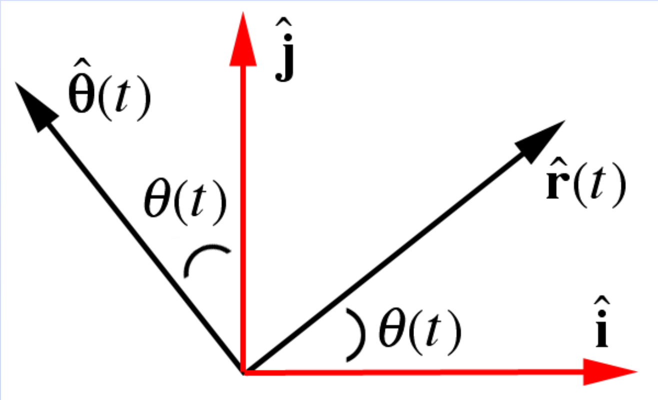



Calculation:

$$\vec{\mathbf{v}}(t) = \frac{d\vec{\mathbf{r}}(t)}{dt} = \frac{d}{dt}\Big(r(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}})\Big)$$

$$= r\left(-\sin\theta(t)\frac{d\theta}{dt}\,\hat{\mathbf{i}} + \cos\theta(t)\frac{d\theta}{dt}\,\hat{\mathbf{j}}\right)$$

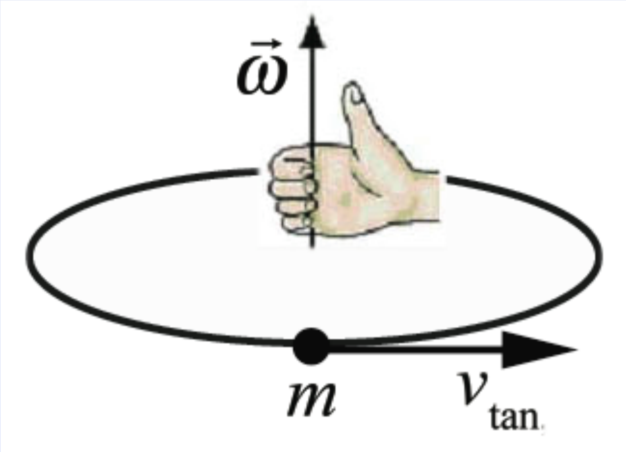
$$= r\frac{d\theta}{dt}(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}) = r\frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t)$$



# Fixed Axis Rotation: Angular Velocity

Angle variable $\theta$

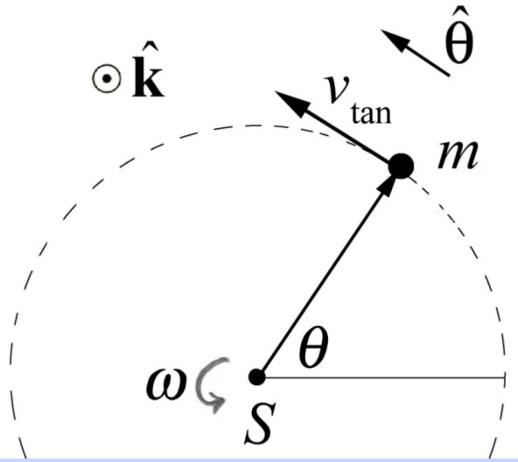
SI unit: $[\mathrm{rad}]$




Angular velocity $\vec{\boldsymbol{\omega}} \equiv \omega_z\, \hat{\mathbf{k}} \equiv (d\theta / dt)\, \hat{\mathbf{k}}$

SI unit: $\left[\mathrm{rad} \cdot \mathrm{s}^{-1}\right]$

Component: $\omega_z \equiv d\theta / dt$




Magnitude $\omega \equiv \left|\omega_z\right| \equiv \left|d\theta / dt\right|$

Direction

$$\omega_z \equiv d\theta / dt > 0, \text{ direction } +\hat{\mathbf{k}}$$

$$\omega_z \equiv d\theta / dt < 0, \text{ direction } -\hat{\mathbf{k}}$$

# Circular Motion: Constant Speed, Period, and Frequency

In one period the object travels a distance equal to the circumference:

$$s = 2\pi R = vT$$

Period: the amount of time to complete one circular orbit of radius R

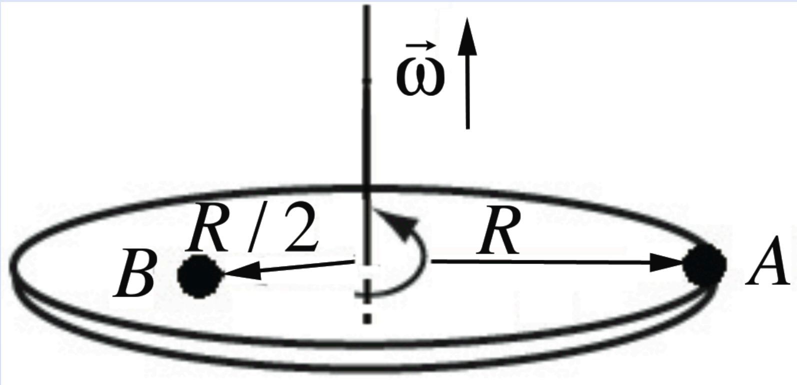
$$\boxed{T = \frac{2\pi R}{v} = \frac{2\pi R}{R\omega} = \frac{2\pi}{\omega}}$$

Frequency is the inverse of the period:

$$\boxed{f = \frac{1}{T} = \frac{\omega}{2\pi} \quad \text{(units: } \mathrm{s}^{-1} \text{ or Hz)}}$$

# Concept Question: Angular Speed

Object A sits at the outer edge (rim) of a merry-go-round, and object B sits halfway between the rim and the axis of rotation. The merry-go-round makes a complete revolution once every thirty seconds. The magnitude of the angular velocity of Object B is

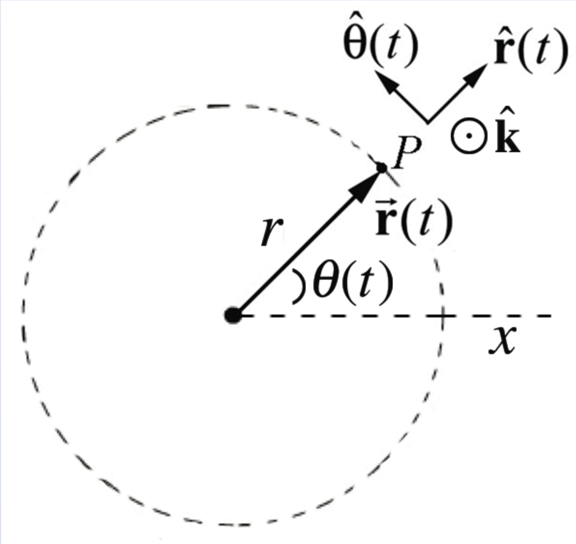



1. half the magnitude of the angular velocity of Object A .
2. the same as the magnitude of the angular velocity of Object A .
3. twice the the magnitude of the angular velocity of Object A .
4. impossible to determine.

# Table Problem: Angular Velocity

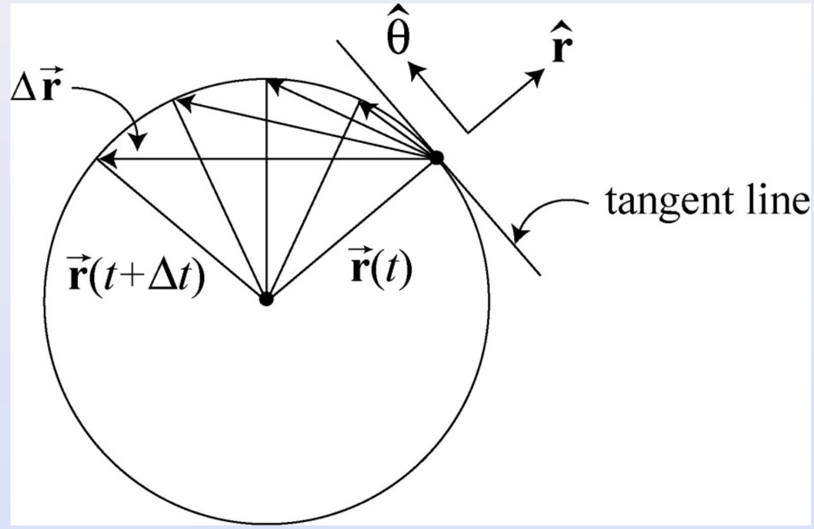
A particle is moving in a circle of radius *R*. At $t = 0$, it is located on the *x*-axis. The angle the particle makes with the positive *x*-axis is given by

$$\theta(t) = At - Bt^3$$



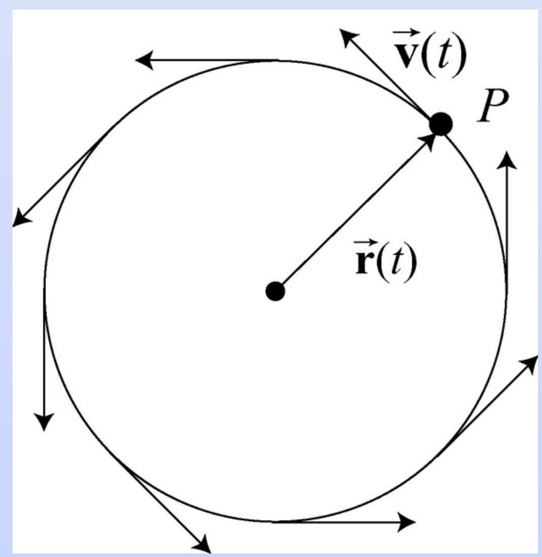

where *A* and *B* are positive constants. Determine the (a) angular velocity vector, and (b) the velocity vector (express your answers in polar coordinates). (c) At what time $t = t_1$ is the angular velocity zero? (d)What is the direction of the angular velocity for (i) $t < t_1$, and (ii) $t > t_1$?

# Direction of Velocity




Sequence of chord $\Delta\vec{r}$ directions approach direction of velocity as $\Delta t$ approaches zero.

The direction of velocity is perpendicular to the direction of the position and tangent to the circular orbit.



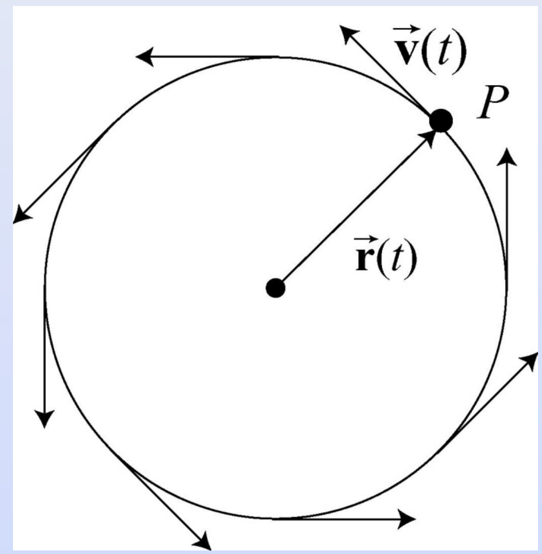

Direction of velocity is constantly changing.

# Circular Motion: Acceleration

# Acceleration and Circular Motion

When an object moves in a circular orbit, the direction of the velocity changes and the speed may change as well.

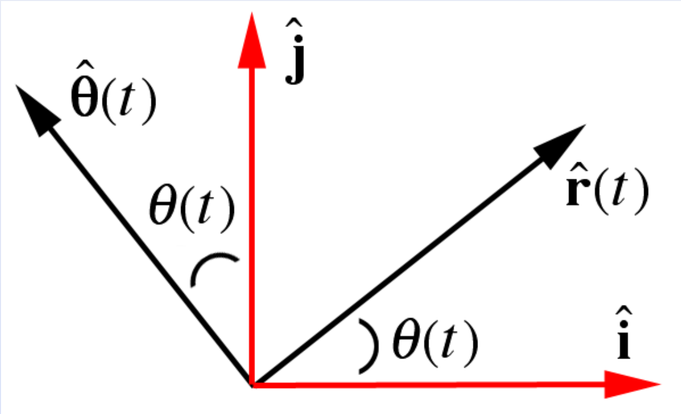
$$\vec{\mathbf{a}}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{v}}}{\Delta t} \equiv \frac{d\vec{\mathbf{v}}}{dt}$$




For circular motion, the acceleration will always have a non-positive radial component ($a_r$) due to the change in direction of velocity, (it may be zero at the instant the velocity is zero).

The acceleration may have a tangential component if the speed changes ($a_t$). When $a_t$ =0, the speed of the object remains constant.

# Math Fact: Derivatives of Unit Vectors in Polar Coordinates




$$\frac{d\hat{\mathbf{r}}(t)}{dt} = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t)$$

$$\frac{d\hat{\boldsymbol{\theta}}(t)}{dt} = \frac{d\theta}{dt}(-\hat{\mathbf{r}}(t))$$

$$\frac{d\hat{\mathbf{r}}(t)}{dt} = \frac{d}{dt}\left(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}\right)$$

$$= -\sin\theta(t)\frac{d\theta}{dt}\,\hat{\mathbf{i}} + \cos\theta(t)\frac{d\theta}{dt}\,\hat{\mathbf{j}}$$

$$= \frac{d\theta}{dt}\left(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\right) = \frac{d\theta}{dt}\hat{\boldsymbol{\theta}}(t)$$

$$\frac{d\hat{\boldsymbol{\theta}}(t)}{dt} = \frac{d}{dt}\left(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\right)$$

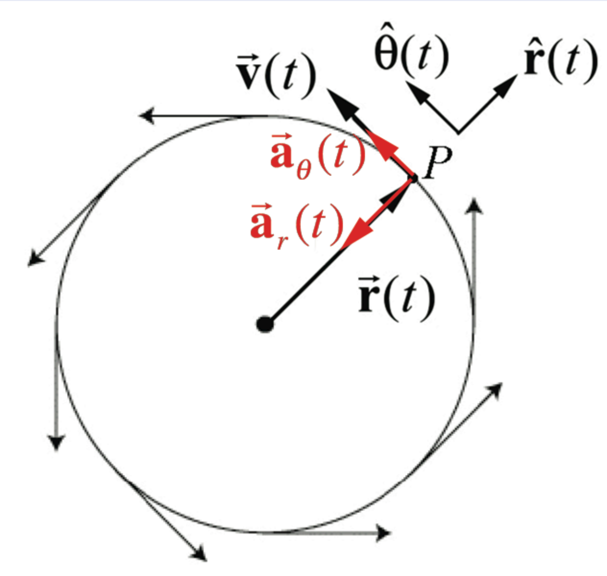
$$= -\cos\theta(t)\frac{d\theta}{dt}\,\hat{\mathbf{i}} - \sin\theta(t)\frac{d\theta}{dt}\,\hat{\mathbf{j}}$$

$$= \frac{d\theta}{dt}\left(-\cos\theta(t)\hat{\mathbf{i}} - \sin\theta(t)\,\hat{\mathbf{j}}\right) = \frac{d\theta}{dt}(-\hat{\mathbf{r}}(t))$$

# Circular Motion: Acceleration

Definition of acceleration

$$\vec{\mathbf{a}}(t) \equiv \frac{d\vec{\mathbf{v}}(t)}{dt} = \frac{d}{dt}\left( r\frac{d\theta}{dt}(t)\hat{\boldsymbol{\theta}}(t) \right)$$




Use product rule

$$\vec{\mathbf{a}}(t) = r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\frac{d\theta}{dt}(t)\frac{d\hat{\boldsymbol{\theta}}(t)}{dt}$$

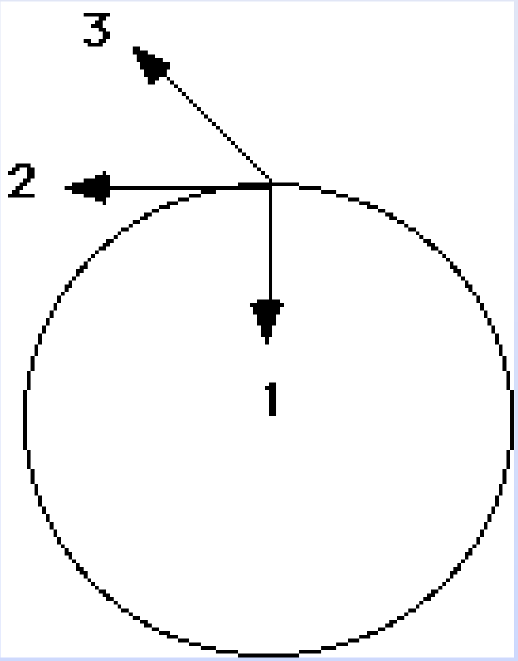
Use differentiation rule

$$\frac{d\hat{\boldsymbol{\theta}}(t)}{dt} = \frac{d\theta}{dt}(-\hat{\mathbf{r}}(t))$$

Result $$\vec{\mathbf{a}}(t) = r\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}}(t) + r\left(\frac{d\theta}{dt}\right)^2(-\hat{\mathbf{r}}(t)) \equiv a_{\theta}\hat{\boldsymbol{\theta}}(t) + a_r\hat{\mathbf{r}}(t)$$

# Concept Question: Circular Motion

As the object speeds up along the circular path in a counterclockwise direction shown below, its acceleration points:

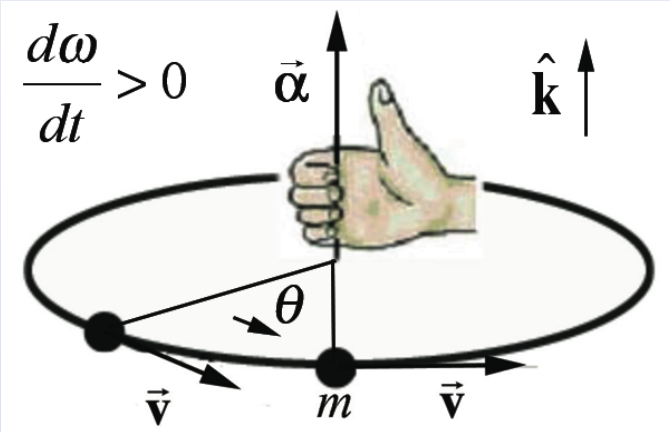



1. toward the center of the circular path.
2. in a direction tangential to the circular path.
3. outward.
4. none of the above.

# Fixed Axis Rotation: Angular Acceleration

Angular acceleration

$$\vec{\boldsymbol{\alpha}} \equiv \alpha_z \, \hat{\mathbf{k}} \equiv (d^2\boldsymbol{\theta} / dt^2) \, \hat{\mathbf{k}}$$




SI unit: $\left[ \mathrm{rad} \cdot \mathrm{s}^{-2} \right]$

Component: $\alpha_z \equiv d^2\boldsymbol{\theta} / dt^2$

Magnitude $\alpha = \left| \alpha_z \right| = \left| d^2\theta / dt^2 \right|$

Direction

$$\alpha_z \equiv d^2\theta / dt^2 > 0, \text{ direction } +\hat{\mathbf{k}}$$

$$\alpha_z \equiv d^2\theta / dt^2 < 0, \text{ direction } -\hat{\mathbf{k}}$$

# Circular Motion: Tangential Acceleration

When the component of the angular velocity is a function of time,

$$\omega_z(t) = \frac{d\theta}{dt}(t)$$

The component of the velocity has a non-zero derivative

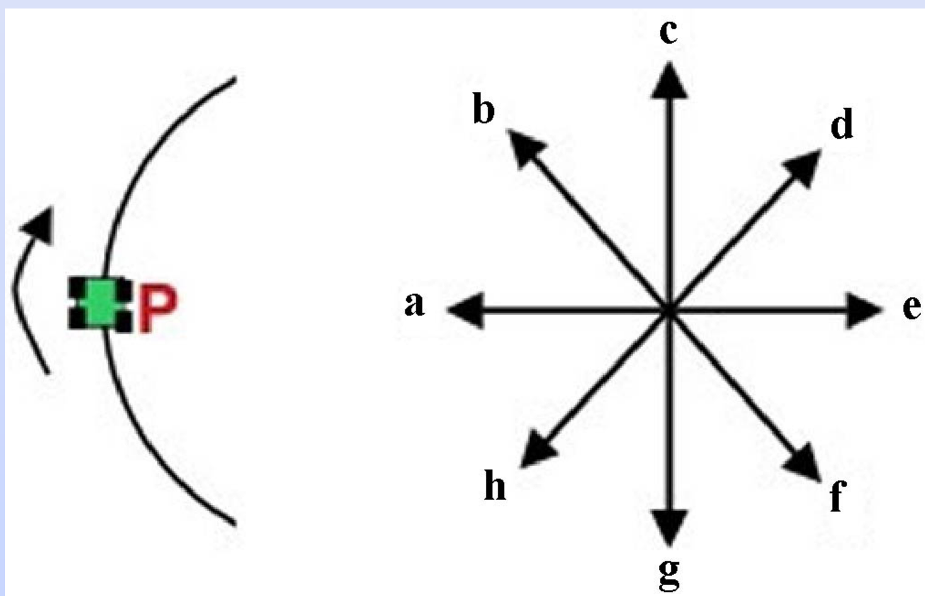
$$\frac{dv_\theta(t)}{dt} = r\frac{d^2\theta}{dt^2}(t)$$

Then the *tangential acceleration* is the time rate of change of the magnitude of the velocity

$$\vec{\mathbf{a}}_\theta(t) = a_\theta(t)\hat{\boldsymbol{\theta}}(t) = r\frac{d^2\theta}{dt^2}(t)\hat{\boldsymbol{\theta}}(t)$$

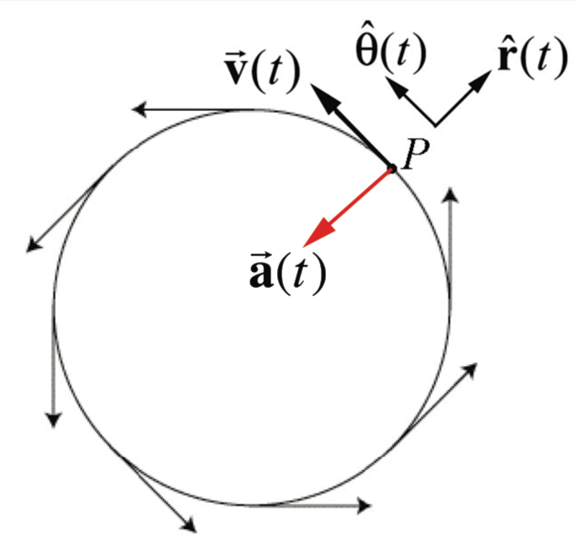
# Concept Question: Cart in a Turn

A golf cart moves around a circular path on a level surface with decreasing speed. Which arrow is closest to the direction of the car's acceleration while passing the point P?



# Constant Speed Circular Motion: Centripetal Acceleration

Position $\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t)$




Angular Speed $\omega \equiv \left| d\theta / dt \right| =$ constant

Velocity $\vec{\mathbf{v}}(t) = v_\theta(t)\,\hat{\boldsymbol{\theta}}(t) = r\dfrac{d\theta}{dt}\,\hat{\boldsymbol{\theta}}(t)$

Speed $v = \left| \vec{\mathbf{v}}(t) \right|$

Angular Acceleration $\alpha \equiv d\omega / dt \;= d^2\theta / dt^2 = 0$

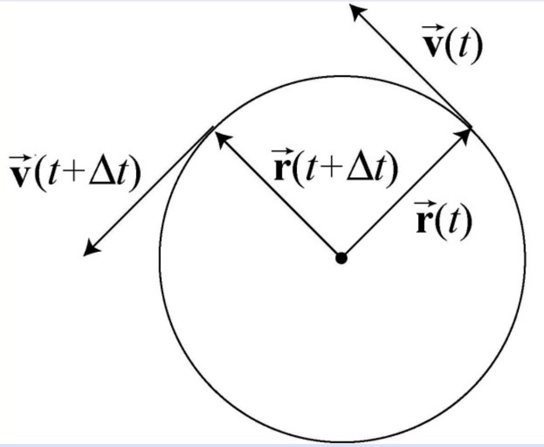
Acceleration $\vec{\mathbf{a}}(t) = a_r(t)\,\hat{\mathbf{r}} = -r\omega^2\hat{\mathbf{r}}(t) = -(v^2 / r)\hat{\mathbf{r}}(t) = -v\omega\hat{\mathbf{r}}(t)$

# Alternative Forms of Magnitude of Centripetal Acceleration

Parameters: speed *v*, angular speed ω, frequency *f*, period *T*

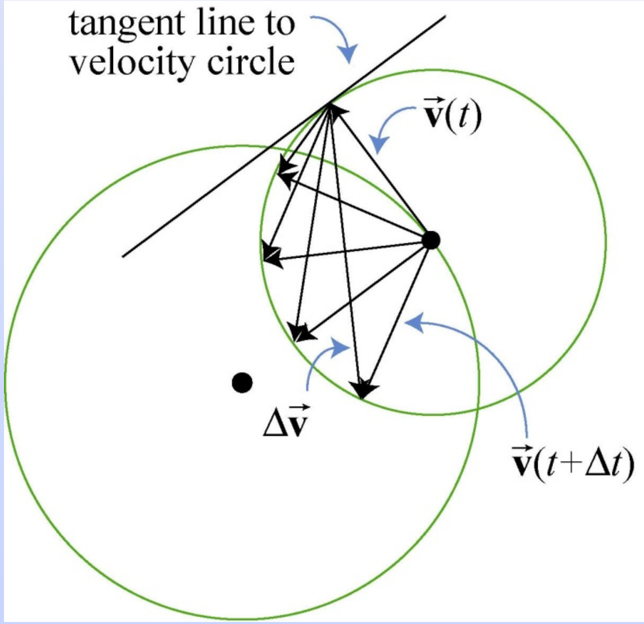
$$\left|a_r\right| = \frac{v^2}{r} = r\omega^2 = r(2\pi f)^2 = \frac{4\pi^2 r}{T^2}$$

# Direction of Radial Acceleration: Uniform Circular Motion




$$\vec{\mathbf{a}}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{\mathbf{v}}}{\Delta t} \equiv \frac{d\vec{\mathbf{v}}}{dt}$$

Sequence of chord directions $\Delta \vec{\mathbf{v}}$ approaches radial inward direction as $\Delta t$ approaches zero




Perpendicular to the velocity vector

Points radially inward

# Concept Question

A car is rounding a circular turn of radius R=200m at constant speed. The magnitude of its centripetal acceleration is 2 $m/s^2$. What is the speed of the car?

1. 400 m/s
2. 20 m/s
3. 100 m/s
4. 10 m/s
5. None of the above.

# Circular Motion: Vector Description

Position $$\vec{\mathbf{r}}(t) = r\,\hat{\mathbf{r}}(t)$$

Component of Angular Velocity $$\omega_z \equiv d\theta / dt$$

Velocity $$\vec{\mathbf{v}} = v_\theta\,\hat{\boldsymbol{\theta}}(t) = r(d\theta / dt)\,\hat{\theta}$$

Component of Angular Acceleration $$\alpha_z \equiv d\omega_z / dt = d^2\theta / dt^2$$

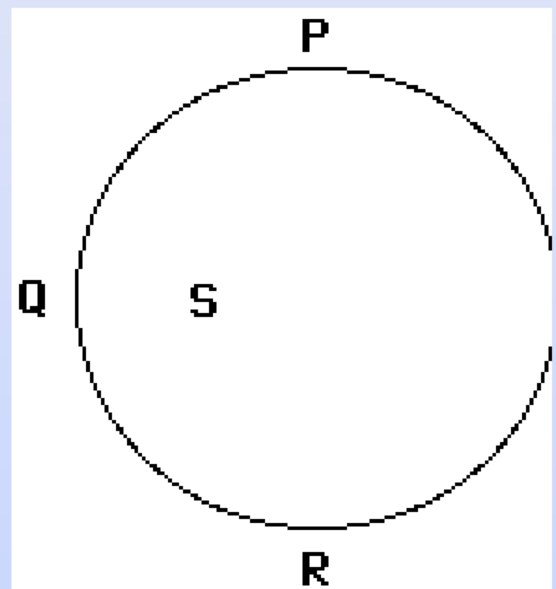
Acceleration $$\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}} + a_\theta\,\hat{\theta}$$

$$a_r = -r(d\theta / dt)^2 = -(v^2 / r), \quad a_\theta = r(d^2\theta / dt^2)$$

# Concept Question: Circular Motion

An object moves counter-clockwise along the circular path shown below. As it moves along the path its acceleration vector continuously points toward point *S*. The object

1. speeds up at *P*, *Q*, and *R*.
2. slows down at *P*, *Q*, and *R*.
3. speeds up at *P* and slows down at *R*.
4. slows down at *P* and speeds up at *R*.
5. speeds up at *Q*.
6. slows down at *Q*.
7. No object can execute such a motion.

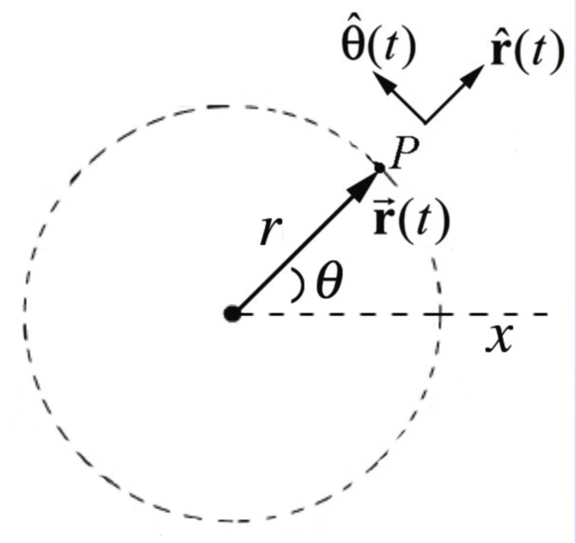


# Table Problem: Circular Motion

A particle is moving in a circle of radius *R*. At $t$ = 0, it is located on the *x*-axis. The angle the particle makes with the positive *x*-axis is given by

$$\theta(t) = At^3 - Bt$$

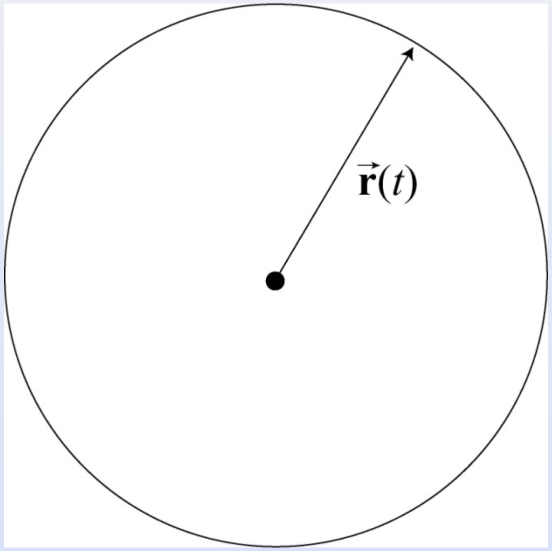
where *A* and *B* are positive constants. Determine the (a) velocity vector, and (b) acceleration vector (express your answers in polar coordinates). (c) At what time is the centripetal acceleration zero?



# Appendix

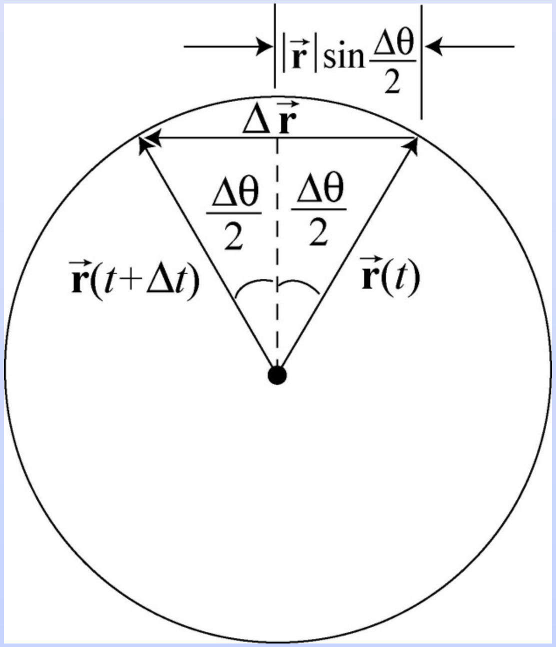
# Position and Displacement

$\vec{\mathbf{r}}(t)$: position vector of an object moving in a circular orbit of radius $R$




$\Delta\vec{\mathbf{r}}(t)$: change in position between time $t$ and time t+Dt

Position vector is changing in direction not in magnitude.




The magnitude of the displacement is the length of the chord of the circle:

$$\boxed{\left|\Delta\vec{\mathbf{r}}\right| = 2R\sin(\Delta\theta/2)}$$

# Small Angle Approximation

Power series expressions for trigonometric functions

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots$$

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \ldots$$

When the angle is small: $\sin\phi \approx \phi, \quad \cos\phi \approx 1$

Using the small angle approximation with $\phi = \Delta\theta / 2$ ,

the magnitude of the displacement is

$$\left|\Delta\vec{\mathbf{r}}\right| = 2R\sin(\Delta\theta / 2) \approx R\,\Delta\theta$$

# Speed and Angular Speed

The speed of the object undergoing circular motion is proportional to the rate of change of the angle with time:

$$v \equiv \left|\vec{\mathbf{v}}\right| = \lim_{\Delta t \to 0} \frac{\left|\Delta \vec{\mathbf{r}}\right|}{\Delta t} = \lim_{\Delta t \to 0} \frac{R\left|\Delta \theta\right|}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\left|\Delta \theta\right|}{\Delta t} = R\left|\frac{d\theta}{dt}\right| = R\omega$$

Angular speed: $\omega = \left|\dfrac{d\theta}{dt}\right| \quad (\text{units: rad} \cdot \text{s}^{-1})$

# Magnitude of Change in Velocity: Circular Motion Solution

Change in velocity:

$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$$

Magnitude of change in velocity:

$$\left|\Delta\vec{\mathbf{v}}\right| = 2v\sin(\Delta\theta/2)$$

Using small angle approximation

$$\boxed{\left|\Delta\vec{\mathbf{v}}\right| \cong v\Delta\theta}$$