Kinetic Energy and Work

8.01

W06D1

Today's Readings:

Chapter 13 The Concept of Energy and Conservation of Energy,

Sections 13.1-13.8

Announcements

Problem Set 4 due Week 6 Tuesday at 9 pm in box outside 26-152

Math Review Week 6 Tuesday at 9 pm in 26-152

Kinetic Energy

- Scalar quantity (reference frame dependent)

$$\boxed{K = \frac{1}{2}mv^2 \geq 0}$$

- SI unit is joule:

$$1\,\mathrm{J} \equiv 1\,\mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}^2$$

- Change in kinetic energy:

$$\boxed{\Delta K = \frac{1}{2}mv_f^2 - \frac{1}{2}mv_0^2 = \frac{1}{2}m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2}m(v_{x,0}^2 + v_{y,0}^2 + v_{z,0}^2)}$$

Momentum and Kinetic Energy: Single Particle

Kinetic energy and momentum for a single particle are related by

$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m}$$

Concept Question: Pushing Carts

Consider two carts, of masses *m* and 2*m*, at rest on an air track. If you push one cart for 3 seconds and then the other for the same length of time, exerting equal force on each, the kinetic energy of the light cart is

1) larger than
2) equal to
3) smaller than

the kinetic energy of the heavy car.

Work Done by a Constant Force for One Dimensional Motion

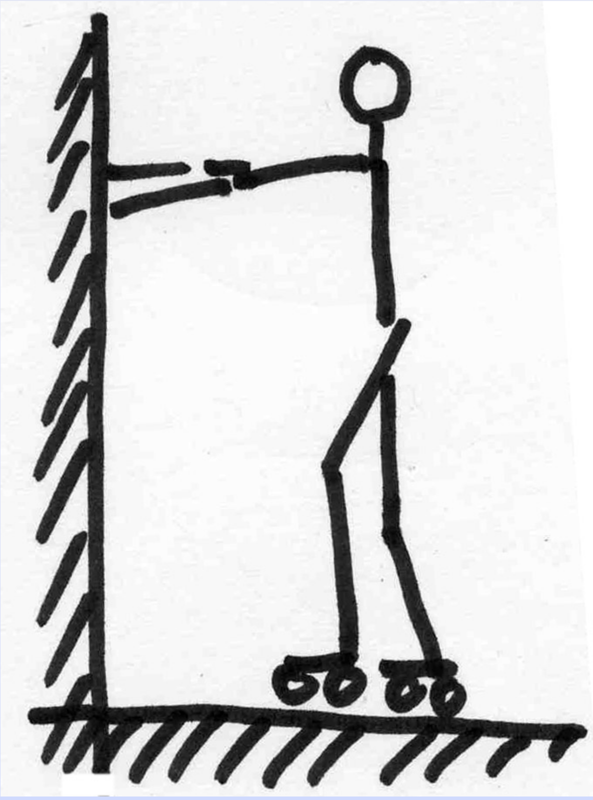
Definition:

The work W done by a constant force with an x-component, F_x, in displacing an object by Δx is equal to the x-component of the force times the displacement:

$$\boxed{W = F_x \Delta x}$$

Concept Q.: Pushing Against a Wall

The work done by the contact force of the wall on the person as the person moves away from the wall is



1. positive.
2. negative.
3. zero.
4. impossible to determine from information given in question and the figure.

Concept Question: Work and Walking

When a person walks, the force of friction between the floor and the person's feet accelerates the person forward. The work done by the friction force is

1. positive.
2. negative.
3. zero.

Worked Example: Work Done by Gravity Near the Surface of the Earth

Consider an object of mass m near the surface of the earth falling directly towards the center of the earth. The gravitational force between the object and the earth is nearly constant. Suppose the object starts from an initial point that is a distance y_0 from the surface of the earth and moves to a final point a distance y_f from the surface of the earth. How much work does the gravitational force do on the object as it falls?

Work done by Non-Constant Force: One Dimensional Motion

(Infinitesimal) work is a scalar

$$\Delta W_i = (F_x)_i \Delta x_i$$

Add up these scalar quantities to get the total work as area under graph of F_x *vs* x :

$$W = \sum_{i=1}^{i=N} \Delta W_i = \sum_{i=1}^{i=N} (F_x)_i \Delta x_i$$

$$\text{As } N \to \infty \text{ and } \left|\Delta x_i\right| \to 0$$

$$W = \lim_{\substack{N\to\infty \\ \Delta x_i \to 0}} \sum_{i=1}^{i=N} (F_x)_i \Delta x_i = \int_{x=x_0}^{x=x_f} F_x dx$$

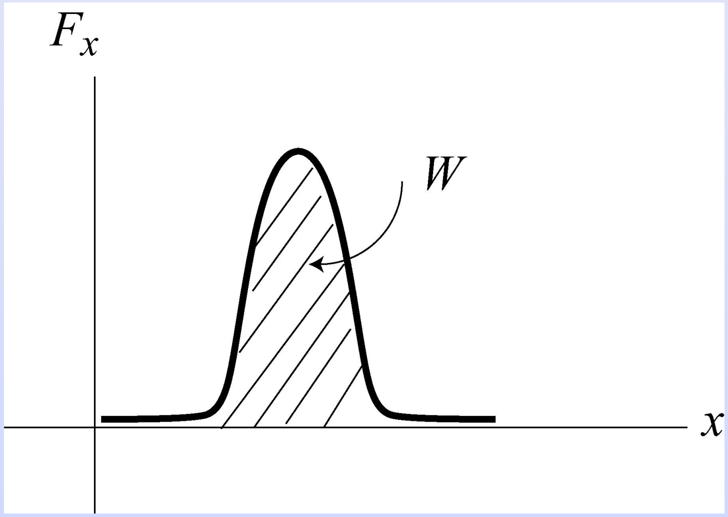
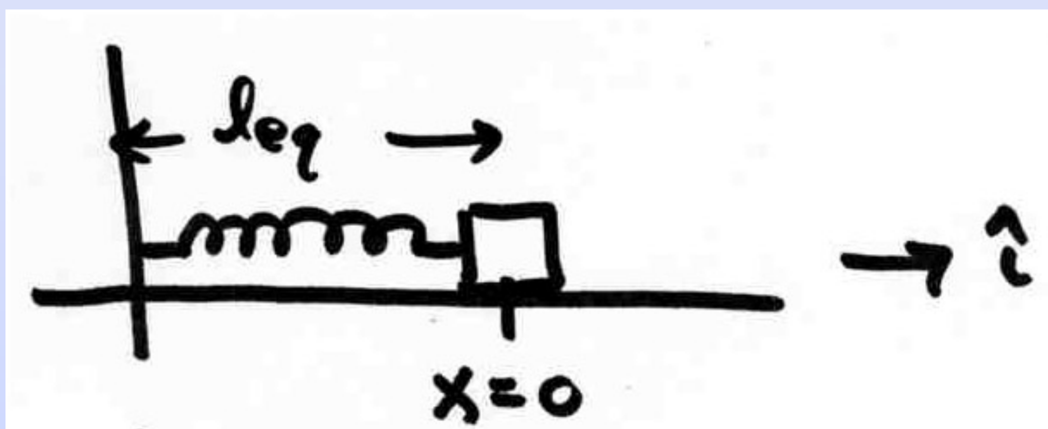


Table Problem: Work Done by the Spring Force

Connect one end of a spring of length l_{eq} with spring constant k to an object resting on a smooth table and fix the other end of the spring to a wall. Stretch the spring until it has length l_i and release the object. How much work does the spring do on the object as a function of $x = l - l_{eq}$, where l is the length of the spring ?



Recall: Integration of Acceleration with Respect to Time

The *x*-component of the acceleration of an object is the derivative of the *x*-component of the velocity

$$a_x \equiv \frac{dv_x}{dt}$$

Therefore the integral of *x*-component of the acceleration with respect to time, is the *x*-component of the velocity

$$\int_{t_0}^{t_f} a_x dt = \int_{t_0}^{t_f} \frac{dv_x}{dt} dt = \int_{v_{x,0}}^{v_{x,f}} dv_x = v_{x,f} - v_{x,0}$$

Integration of Acceleration with Respect to Displacement

The integral of x-component of the acceleration with respect to the displacement of an object, is given by

$$\int_{x_0}^{x_f} a_x dx = \int_{x_0}^{x_f} \frac{dv_x}{dt} dx = \int_{x_0}^{x_f} dv_x \frac{dx}{dt} = \int_{v_{x,0}}^{v_{x,f}} v_x dv_x$$

$$\int_{x_0}^{x_f} a_x dx = \int_{v_{x,0}}^{v_{x,f}} d\left((1/2)v_x{}^2\right) = \frac{1}{2}(v_{x,f}^2 - v_{x,0}^2)$$

Multiply both sides by the mass of the object giving integration formula

$$\int_{x_0}^{x_f} m a_x dx = \int_{v_{x,0}}^{v_{x,f}} d\left((1/2)v_x{}^2\right) = \frac{1}{2} m v_{x,f}^2 - \frac{1}{2} m v_{x,0}^2 = \Delta K$$

Work-Kinetic Energy Theorem One Dimensional Motion

Substitute Newton's Second Law (in one dimension)

$$F_x = ma_x$$

in definition of work integral which then becomes

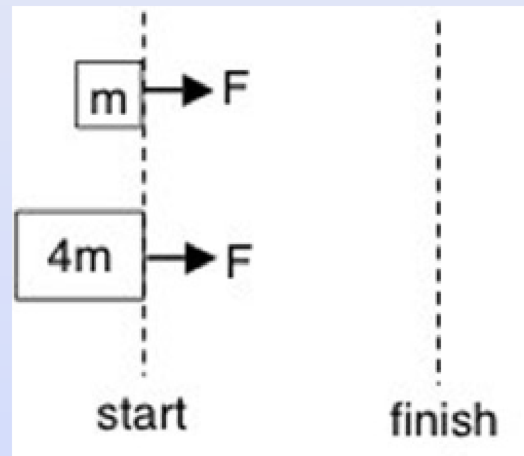
$$W = \int_{x_0}^{x_f} F_x \, dx = \int_{x_0}^{x_f} ma_x \, dx$$

Apply integration formula to get work-kinetic energy theorem

$$W = \int_{x_0}^{x_f} F_x \, dx = \int_{x_0}^{x_f} ma_x \, dx = \Delta K$$

Concept Question

Two objects are pushed on a frictionless surface from a starting line to a finish line with equal constant forces. One object is four times as massive as the other. Both objects are initially at rest. Which of the following statements is true when the objects reach the finish line?



1. The kinetic energies of the two objects are equal.
2. Object of mass 4m has the greater kinetic energy.
3. Object of mass m has the greater kinetic energy.
4. Not information is given to decide.

Concept Question: Work due to Variable Force

A particle starts from rest at x = 0 and moves to $x = L$ under the action of a variable force $F(x)$, which is shown in the figure. What is the particle's kinetic energy at $x = L/2$ and at x = L?

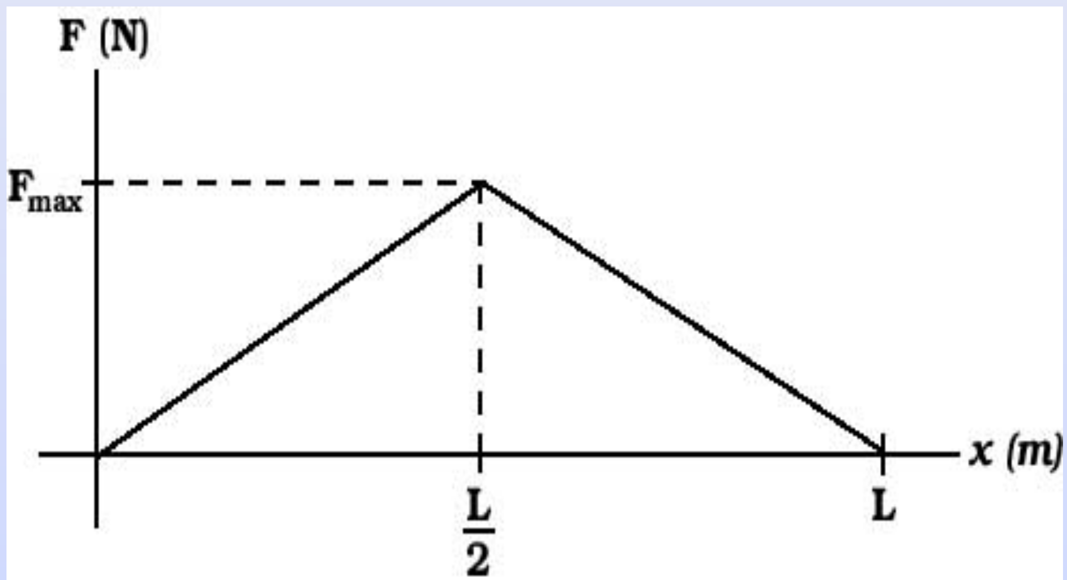
(1) $(F_{max})(L/2)$, $(F_{max})(L)$

(2) $(F_{max})(L/4)$, 0

(3) $(F_{max})(L)$, 0

(4) $(F_{max})(L/4)$, $(F_{max})(L/2)$

(5) $(F_{max})(L/2)$, $(F_{max})(L/4)$



Power

The average power of an applied force is the rate of doing work

$$\overline{P} = \frac{\Delta W}{\Delta t} = \frac{F_{\text{applied},x}\,\Delta x}{\Delta t} = F_{\text{applied},x}\overline{v}_x$$

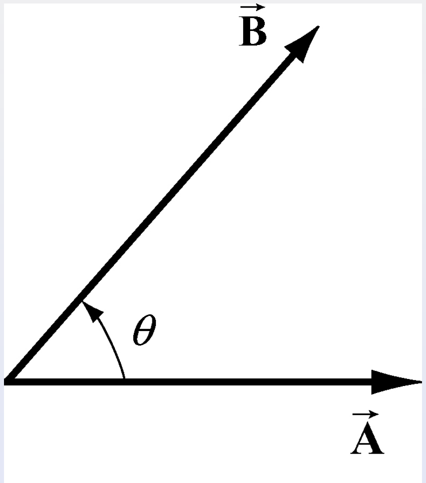
SI units of power: Watts

$$1\,\mathrm{W} \equiv 1\,\mathrm{J/s} = 1\,\mathrm{kg}\cdot\mathrm{m^2/s^3}$$

Instantaneous power

$$P = \lim_{\Delta t \to 0}\frac{\Delta W}{\Delta t} = F_{\text{applied},x}\left(\lim_{\Delta t \to 0}\frac{\Delta x}{\Delta t}\right) = F_{\text{applied},x}\,v_x$$

Scalar Product

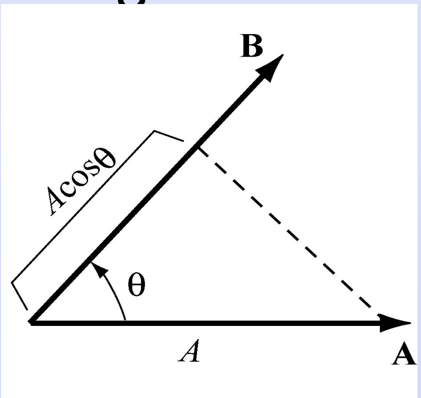


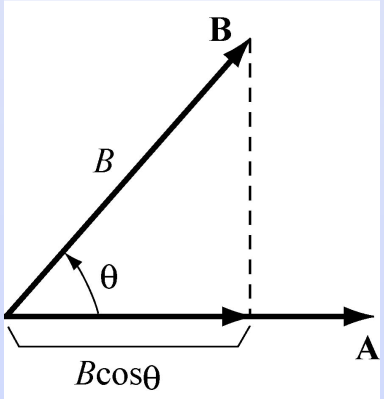
A scalar quantity

Magnitude: $$\boxed{\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|\left|\vec{\mathbf{B}}\right| \cos\theta}$$

The scalar product can be positive, zero, or negative

Two types of projections: the scalar product is the parallel component of one vector with respect to the second vector times the magnitude of the second vector





$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = A_{\parallel}\left|\vec{\mathbf{B}}\right|$$

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = \left|\vec{\mathbf{A}}\right|(\cos\theta)\left|\vec{\mathbf{B}}\right| = \left|\vec{\mathbf{A}}\right| B_{\parallel}$$

Scalar Product: Unit Vectors in Cartesian Coordinates

For unit vectors $\hat{\mathbf{i}}, \hat{\mathbf{j}}$

We have

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = |\hat{\mathbf{i}}||\hat{\mathbf{i}}|\cos(0) = 1$$

$$\hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\cos(\pi/2) = 0$$

Generally:

$$\boxed{\begin{aligned} \hat{\mathbf{i}} \cdot \hat{\mathbf{i}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{k}} \cdot \hat{\mathbf{k}} = 1 \\ \hat{\mathbf{i}} \cdot \hat{\mathbf{j}} = \hat{\mathbf{i}} \cdot \hat{\mathbf{k}} = \hat{\mathbf{j}} \cdot \hat{\mathbf{k}} = 0 \end{aligned}}$$

Scalar Product: Cartesian Coordinates

$$\vec{\mathbf{A}} = A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}, \quad \vec{\mathbf{B}} = B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}}$$

Then

$$\vec{\mathbf{A}} \cdot \vec{\mathbf{B}} = (A_x \hat{\mathbf{i}} + A_y \hat{\mathbf{j}} + A_z \hat{\mathbf{k}}) \cdot (B_x \hat{\mathbf{i}} + B_y \hat{\mathbf{j}} + B_z \hat{\mathbf{k}})$$

$$= A_x B_x + A_y B_y + A_z B_z$$

Kinetic Energy and Scalar Product

Velocity

$$\boxed{\vec{\mathbf{v}} = v_x \hat{\mathbf{i}} + v_y \hat{\mathbf{j}} + v_z \hat{\mathbf{k}}}$$

Kinetic Energy:

$$K = \frac{1}{2} m(\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}) = \frac{1}{2} m(v_x^2 + v_y^2 + v_z^2) \geq 0$$

Change in kinetic energy:

$$\boxed{\begin{aligned}&\Delta K = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_0^2 = \frac{1}{2} m(\vec{\mathbf{v}}_f \cdot \vec{\mathbf{v}}_f) - \frac{1}{2} m(\vec{\mathbf{v}}_0 \cdot \vec{\mathbf{v}}_0)\\ &\frac{1}{2} m(v_{x,f}^2 + v_{y,f}^2 + v_{z,f}^2) - \frac{1}{2} m(v_{x,0}^2 + v_{y,0}^2 + v_{z,0}^2)\end{aligned}}$$

Work Done by a Constant Force

Definition: Work

The work done by a constant force $\vec{\mathbf{F}}$ on an object is equal to the component of the force in the direction of the displacement times the magnitude of the displacement:

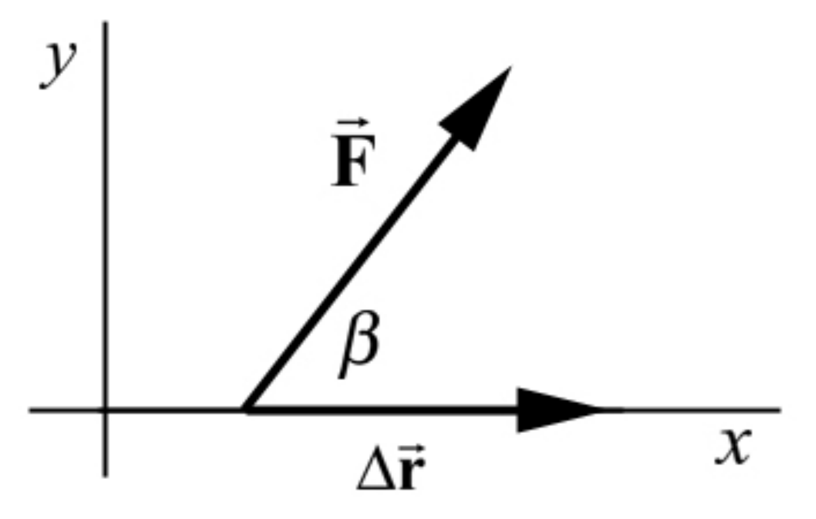
$$\boxed{W = \vec{\mathbf{F}} \cdot \Delta\vec{\mathbf{r}} = \left|\vec{\mathbf{F}}\right| \left|\Delta\vec{\mathbf{r}}\right| \cos\theta = \left|\vec{\mathbf{F}}\right| \cos\theta \left|\Delta\vec{\mathbf{r}}\right| = F_{\|} \left|\Delta\vec{\mathbf{r}}\right|}$$

Note that the component of the force in the direction of the displacement can be positive, zero, or negative so the work may be positive, zero, or negative

Worked Example: Work Done by a Constant Force in Two Dimensions

Force exerted on the object:

$$\vec{\mathbf{F}} = F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}}$$



Components:

$$F_x = F\cos\beta \qquad F_y = F\sin\beta$$

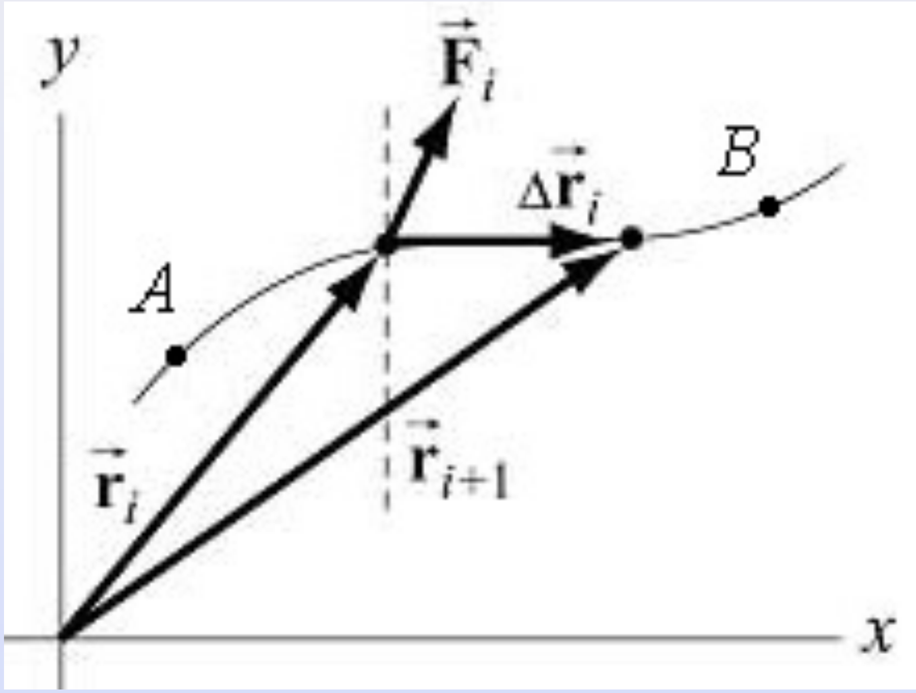
Consider an object undergoing displacement:

$$\Delta\vec{\mathbf{r}} = \Delta x\,\hat{\mathbf{i}}$$

Work done by force on object:

$$W = \vec{\mathbf{F}}\cdot\Delta\vec{\mathbf{r}} = F\Delta x\cos\beta = (F_x\,\hat{\mathbf{i}} + F_y\,\hat{\mathbf{j}})\cdot(\Delta x\,\hat{\mathbf{i}}) = F_x\,\Delta x$$

Work Done Along an Arbitrary Path



Work done by force for small displacement

$$\Delta W_i = \vec{\mathbf{F}}_i \cdot \Delta\vec{\mathbf{r}}_i$$

Work done by force along path from A to B

$$W_{AB} = \lim_{\substack{N\to\infty \\ |\Delta\vec{\mathbf{r}}_i|\to 0}} \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \cdot \Delta\vec{\mathbf{r}}_i \equiv \int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$

Work-Energy Theorem in Three-Dimensions

As you will show in the problem set, the one dimensional work-kinetic energy theorem generalizes to three dimensions

$$W_{AB} = \int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \int_A^B m\vec{\mathbf{a}} \cdot d\vec{\mathbf{r}} = \int_A^B m\frac{d\vec{\mathbf{v}}}{dt} \cdot d\vec{\mathbf{r}} = \int_A^B m\, d\vec{\mathbf{v}} \cdot \frac{d\vec{\mathbf{r}}}{dt} = \int_A^B m\, d\vec{\mathbf{v}} \cdot \vec{\mathbf{v}}$$

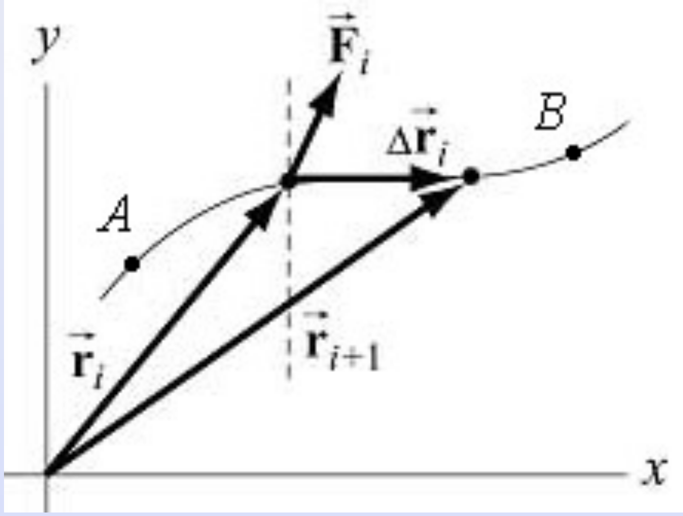
$$= \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 = K_B - K_A$$

$$= \Delta K$$

Work: Path Dependent Line Integral

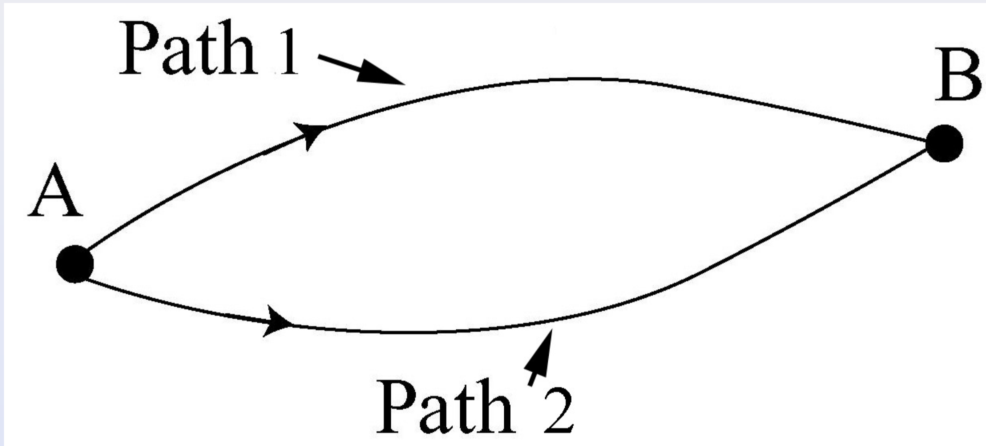
Work done by force along path from A to B

$$W_{AB} = \lim_{\substack{N \to \infty \\ |\Delta\vec{\mathbf{r}}_i| \to 0}} \sum_{i=1}^{i=N} \vec{\mathbf{F}}_i \cdot \Delta\vec{\mathbf{r}}_i \equiv \int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}}$$



In order to calculate the line integral, in principle, requires a knowledge of the path. However we will consider an important class of forces in which the work line integral is independent of the path and only depends on the starting and end points

Conservative Forces



Definition: Conservative Force If the work done by a force in moving an object from point A to point B is independent of the path (1 or 2),

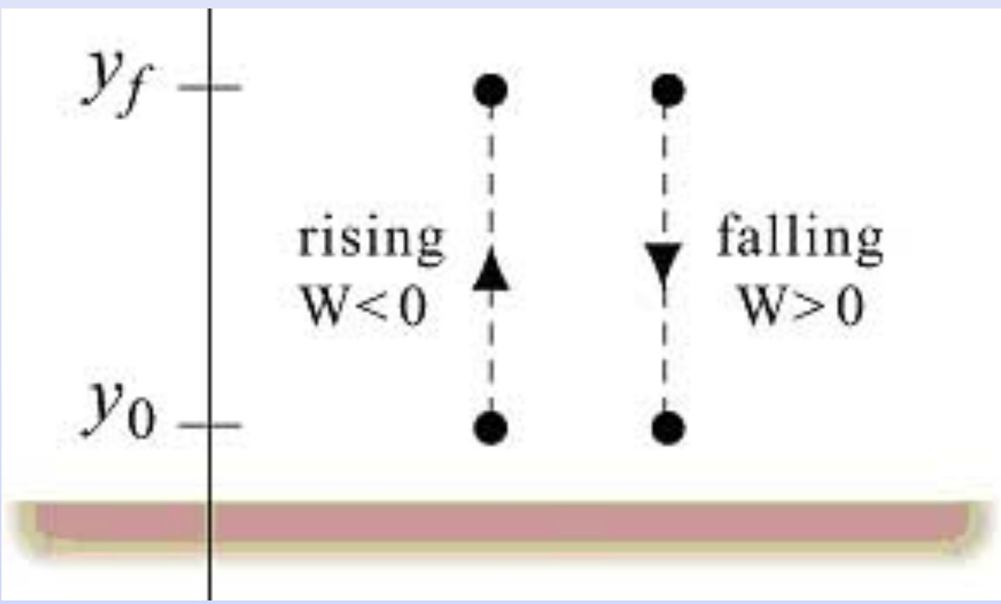
$$W_c \equiv \int_A^B \vec{\mathbf{F}}_{\mathrm{c}} \cdot d\vec{\mathbf{r}} \qquad \text{(path independent)}$$

then the force is called a *conservative force* which we denote by $\vec{\mathbf{F}}_{\mathrm{c}}$. Then the work done only depends on the location of the points A and B.

Example: Gravitational Force

Consider the motion of an object under the influence of a gravitational force near the surface of the earth

The work done by gravity depends only on the change in the vertical position



$$\boxed{W_g = F_g \Delta y = -mg\Delta y}$$

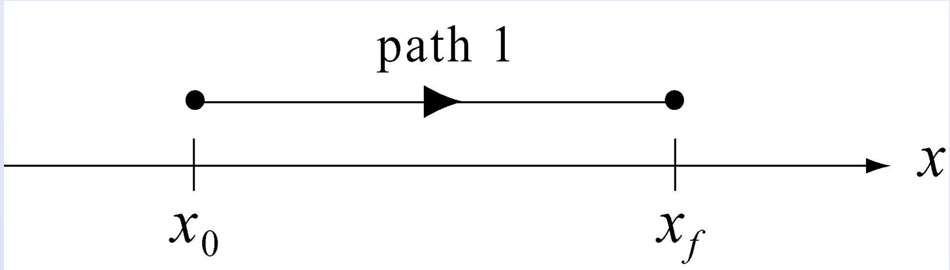
Non-Conservative Forces

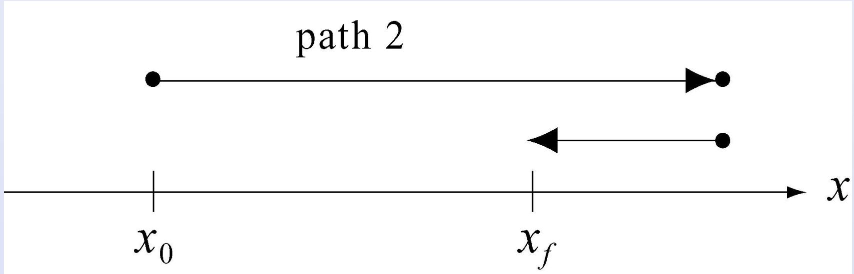
Definition: Non-conservative force Whenever the work done by a force in moving an object from an initial point to a final point depends on the path, then the force is called a *non-conservative force* $\vec{\mathbf{F}}_{nc}$ and the work done is called *non-conservative work*

$$W_{nc} \equiv \int_A^B \vec{\mathbf{F}}_{nc} \cdot d\vec{\mathbf{r}}$$

Non-Conservative Forces

Work done on the object by the force depends on the path taken by the object



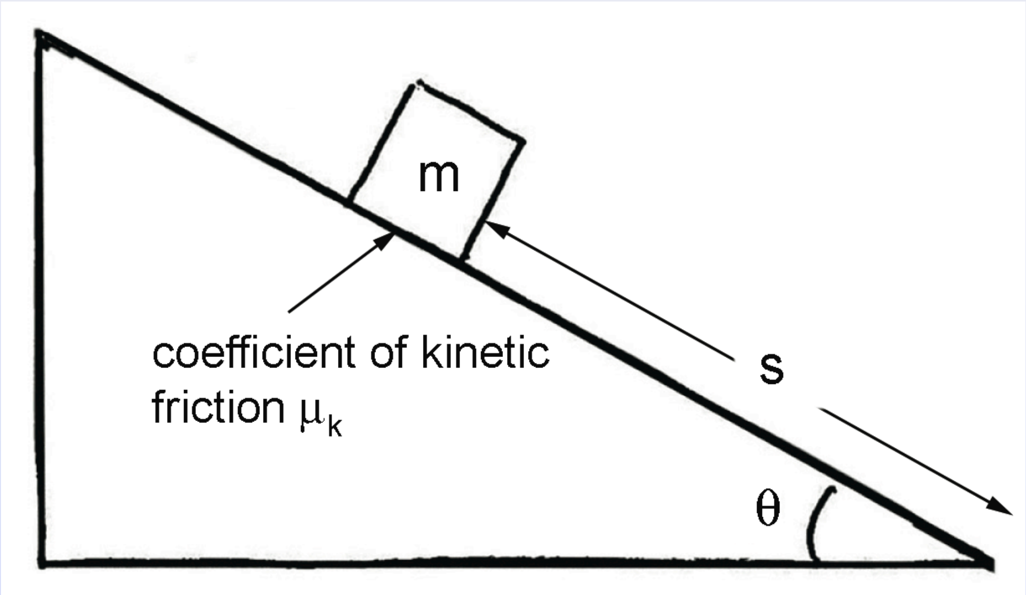


Example: friction on an object moving on a level surface

$$F_{\text{friction}} = \mu_k N$$

$$W_{\text{friction}} = -F_{\text{friction}} \Delta x = -\mu_k N \Delta x < 0$$

Table Problem: Work Constant Forces and Scalar Product



An object of mass m, starting from rest, slides down an inclined plane of length s. The plane is inclined by an angle of θ to the ground. The coefficient of kinetic friction is μ_k. What is the kinetic energy of the object after it slides down the inclined plane a distance s?