Rigid Bodies: Rotational & Translational Motion Rolling without Slipping

8.01
W11D1

Announcements

Sunday Tutoring in 26-152 from 1-5 pm

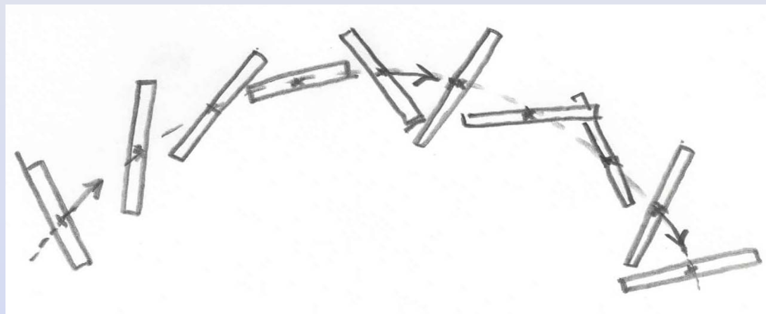
Problem Set 8 due Week 11 Tuesday at 9 pm in box outside 26-152

No Math Review Week 11

Exam 3 Tuesday Nov 26 7:30 to 9:30 pm
Conflict Exam 3 Wednesday Nov 27 8 am to 10 am, 10 am to 12 noon

Nov 27 Drop Date

Demo: Rotation and Translation of a Rigid Body



Thrown Rigid Rod

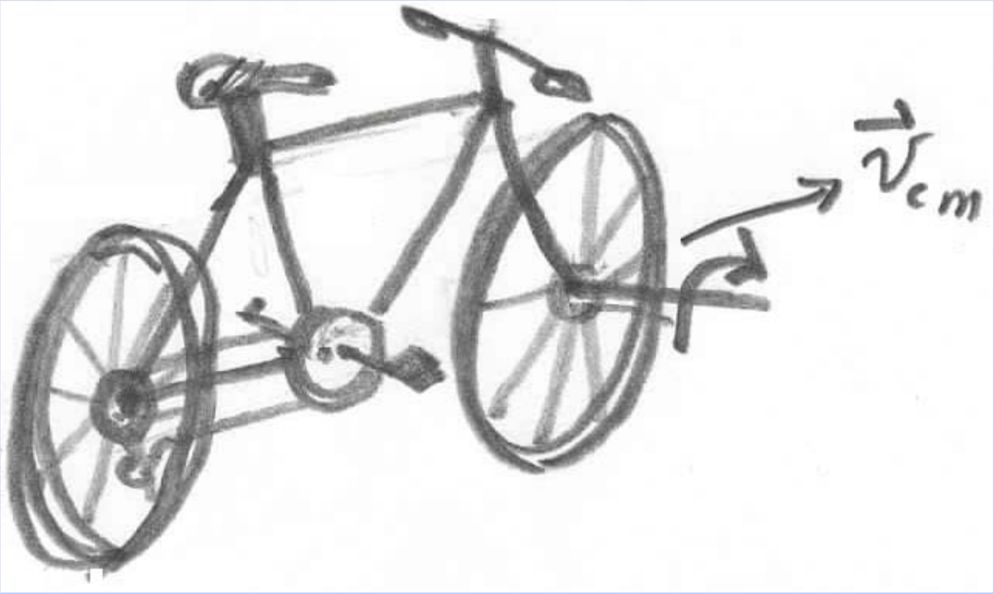
Translational Motion: the gravitational external force **acts on** center-of-mass

$$\vec{\mathbf{F}}^{\text{ext}} = \frac{d\vec{\mathbf{p}}^{\text{sys}}}{dt} = m^{\text{total}} \frac{d\vec{\mathbf{V}}_{\text{cm}}}{dt} = m^{\text{total}} \vec{\mathbf{A}}_{\text{cm}}$$

Rotational Motion: object rotates **about** center-of-mass. Note that the center-of-mass may be accelerating

Rotation about a moving axis

For straight line motion, the bicycle wheel rotates about a fixed direction and center of mass is translating



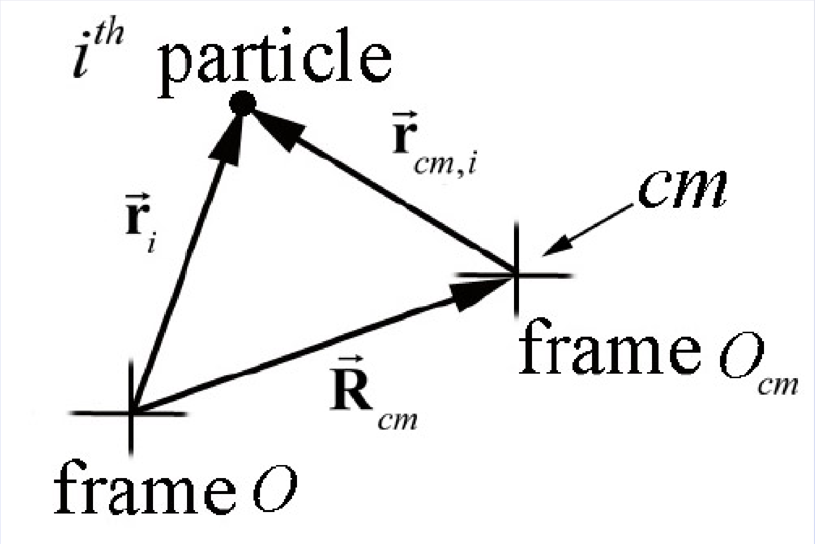
Demo: Bicycle Wheel

Constraint conditions and rolling without slipping

Center of Mass Reference Frame

Frame *O:* At rest with respect to ground

Frame O_{cm}: Origin located at center of mass



Position vectors in different frames:

$$\vec{\mathbf{r}}_i = \vec{\mathbf{r}}_{\mathrm{cm},i} + \vec{\mathbf{R}}_{cm} \qquad \vec{\mathbf{r}}_{\mathrm{cm},i} = \vec{\mathbf{r}}_i - \vec{\mathbf{R}}_{cm}$$

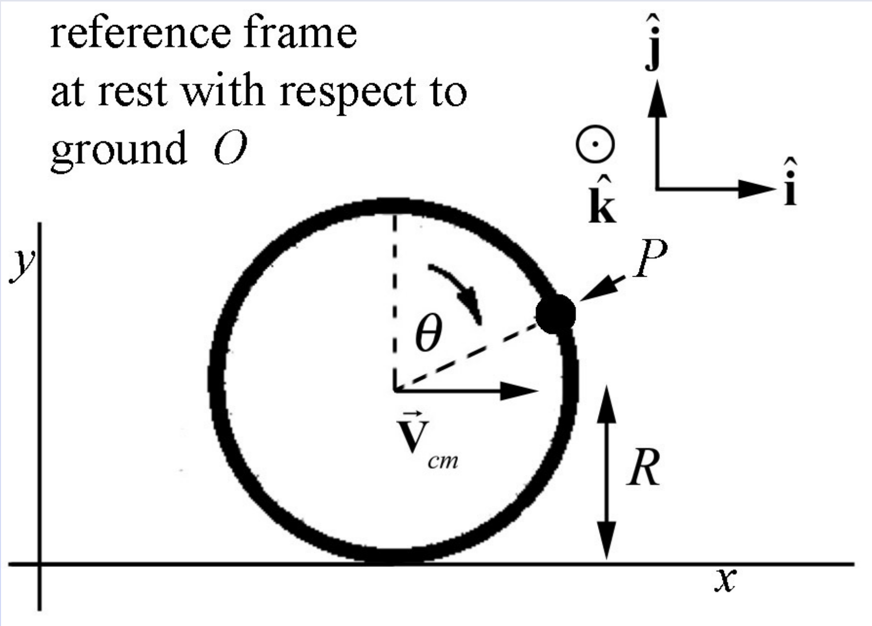
Relative velocity between the two reference frames

$$\vec{\mathbf{V}}_{cm} = d\vec{\mathbf{R}}_{cm} / dt \qquad \vec{\mathbf{A}}_{cm} = d\vec{\mathbf{V}}_{cm} / dt = \vec{\mathbf{0}}$$

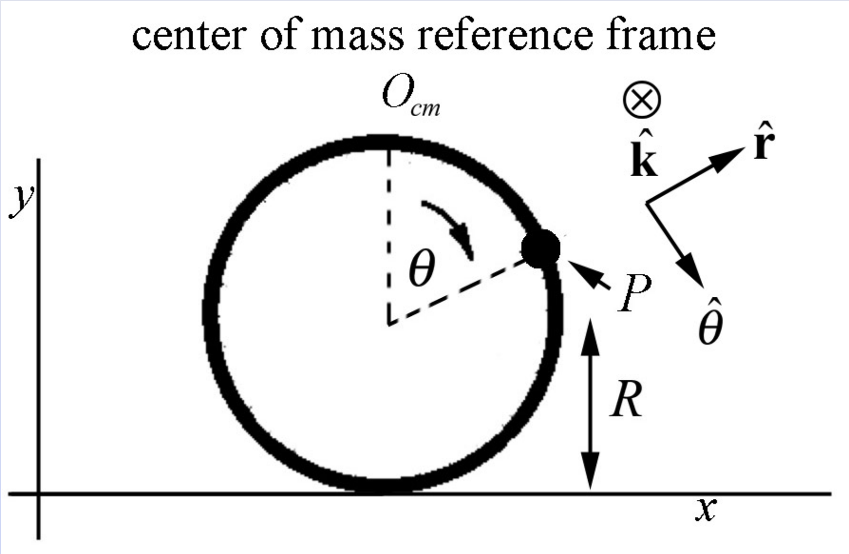
Law of addition of velocities:

$$\vec{\mathbf{v}}_i = \vec{\mathbf{v}}_{\mathrm{cm},i} + \vec{\mathbf{V}}_{cm} \qquad \vec{\mathbf{v}}_{\mathrm{cm},i} = \vec{\mathbf{v}}_i - \vec{\mathbf{V}}_{cm}$$

Rolling Bicycle Wheel



Reference frame fixed to ground



Center of mass reference frame

Motion of point *P* on rim of rolling bicycle wheel

Relative velocity of point *P* on rim: $\vec{\mathbf{v}}_P = \vec{\mathbf{v}}_{\mathrm{cm},P} + \vec{\mathbf{V}}_{cm}$

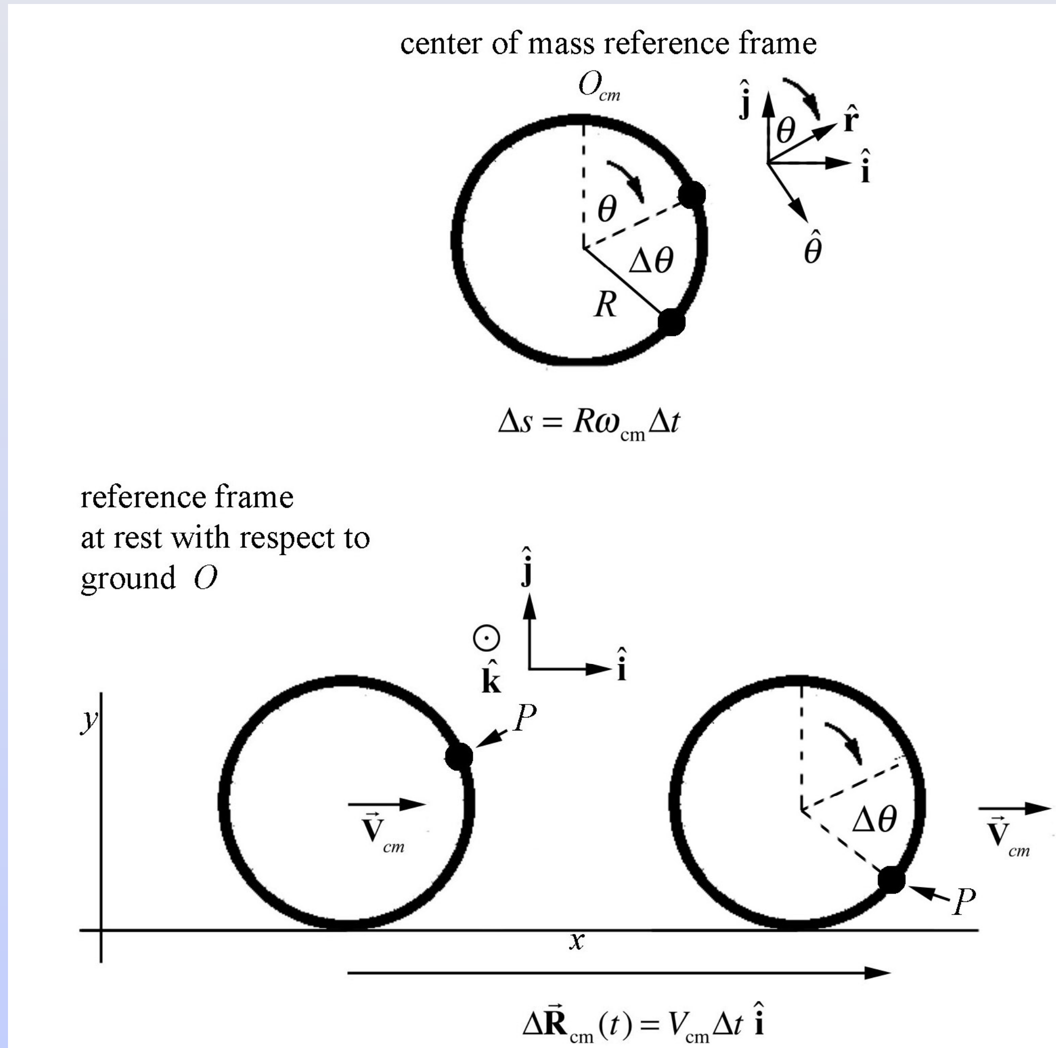
Rolling Bicycle Wheel

Distance traveled in center of mass reference frame of point P on rim in time Δt:

$$\Delta s = R\Delta\theta = R\omega_{\text{cm}}\Delta t$$

Distance traveled in ground fixed reference frame of point P on rim in time Δt:

$$\Delta X_{\text{cm}} = V_{\text{cm}}\Delta t$$



Rolling Bicycle Wheel: Constraint Relations

Rolling without slipping:

$$\Delta s = \Delta X_{\text{cm}}$$

$$R\omega_{\text{cm}} = V_{\text{cm}}$$

Rolling and Skidding:

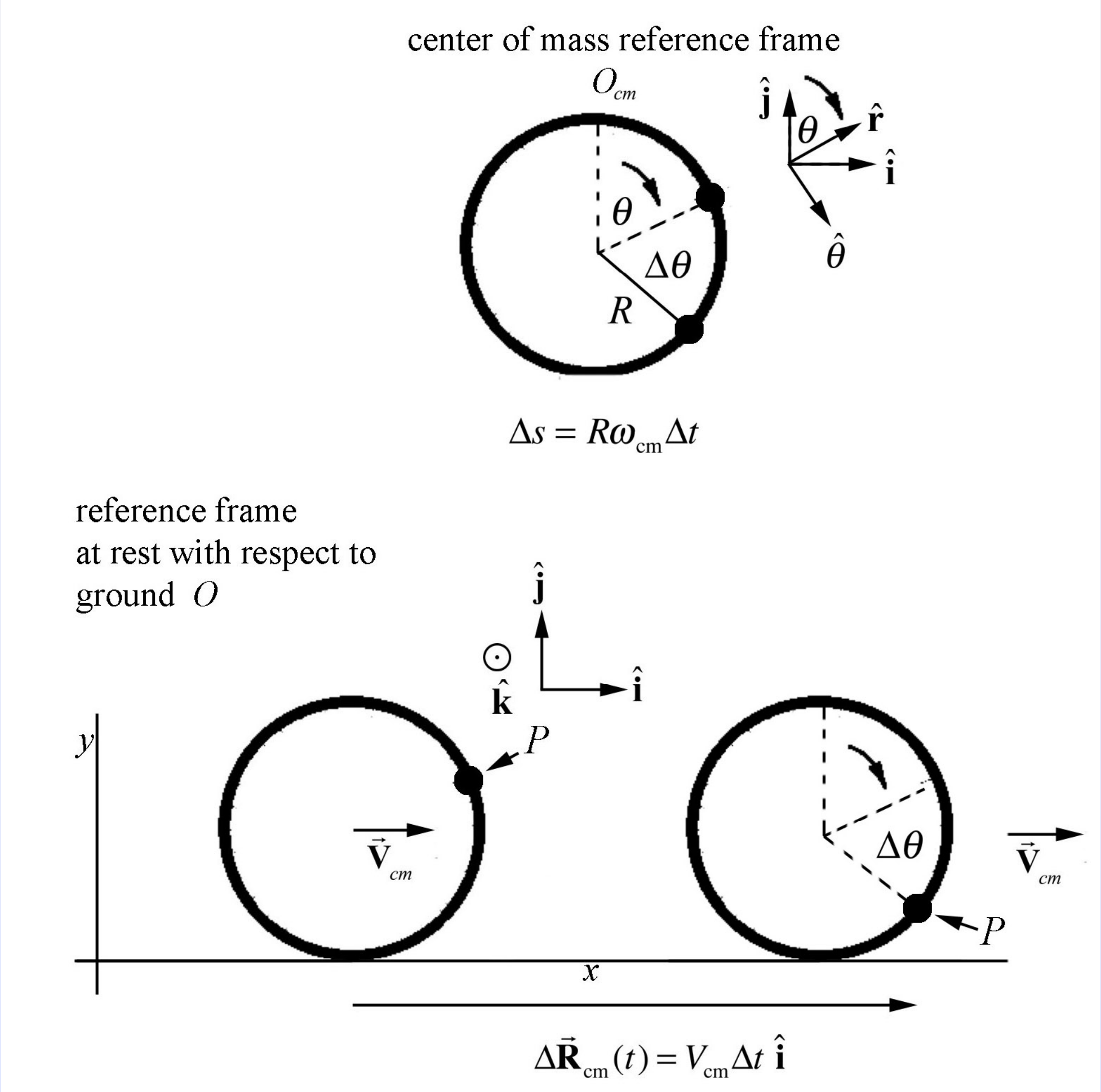
$$\Delta s < \Delta X_{\text{cm}}$$

$$R\omega_{\text{cm}} < V_{\text{cm}}$$

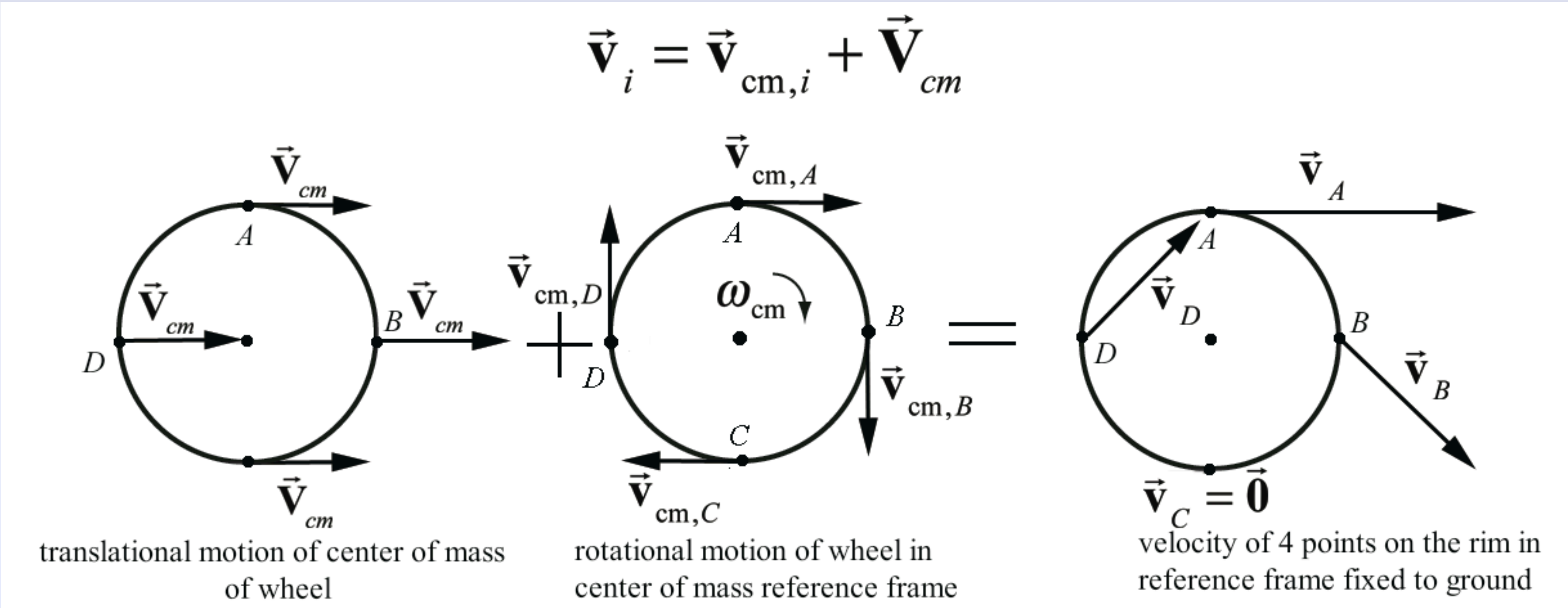
Rolling and Slipping:

$$\Delta s > \Delta X_{\text{cm}}$$

$$R\omega_{\text{cm}} > V_{\text{cm}}$$



Rolling Without Slipping



The velocity of the point on the rim that is in contact with the ground is zero in the reference frame fixed to the ground.

Kinetic Energy of Rotation and Translation

Kinetic energy of rotation about center-of-mass

$$K_{\mathrm{rot}} = \frac{1}{2} I_{\mathrm{cm}} \omega_{\mathrm{cm}}^2$$

Translational kinetic energy

$$K_{\mathrm{trans}} = \frac{1}{2} m v_{\mathrm{cm}}^2$$

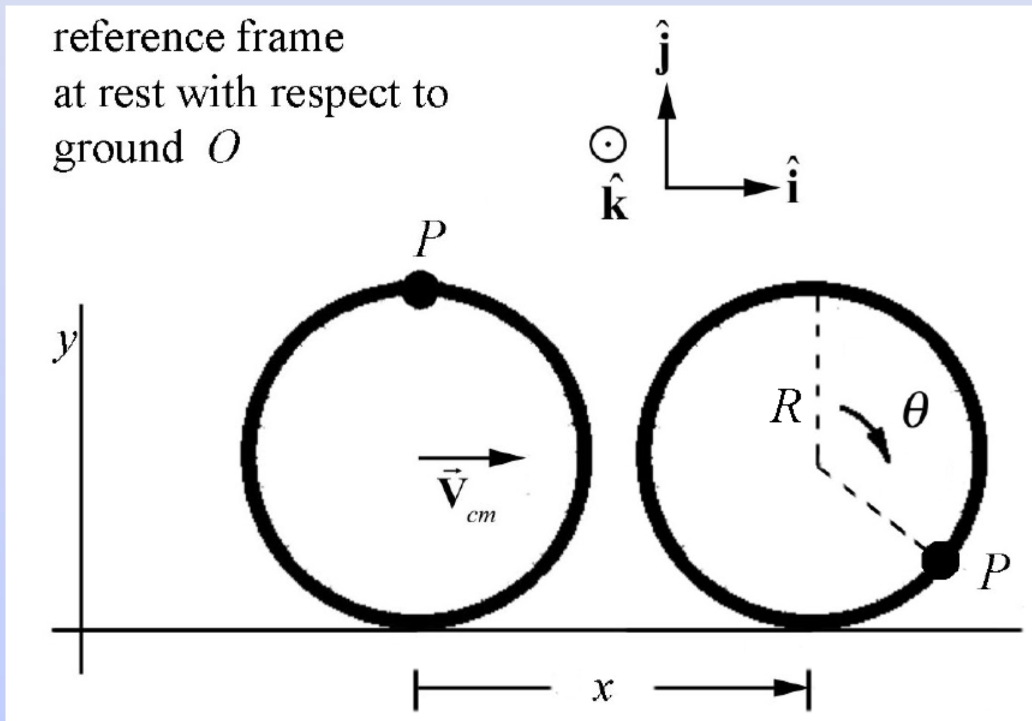
Kinetic energy is sum

$$K = K_{\mathrm{trans}} + K_{\mathrm{rot}} = \frac{1}{2} m v_{\mathrm{cm}}^2 + \frac{1}{2} I_{\mathrm{cm}} \omega_{cm}^2$$

Concept Question: Rolling Without Slipping

If a wheel of radius R rolls without slipping through an angle θ, what is the relationship between the distance the wheel rolls, x, and the product Rθ?

1. $x > R\theta$.
2. $x = R\theta$.
3. $x < R\theta$.



Concept Q. Answer : Rolling Without Slipping

Answer 2. Rolling without slipping condition, x = Rθ.

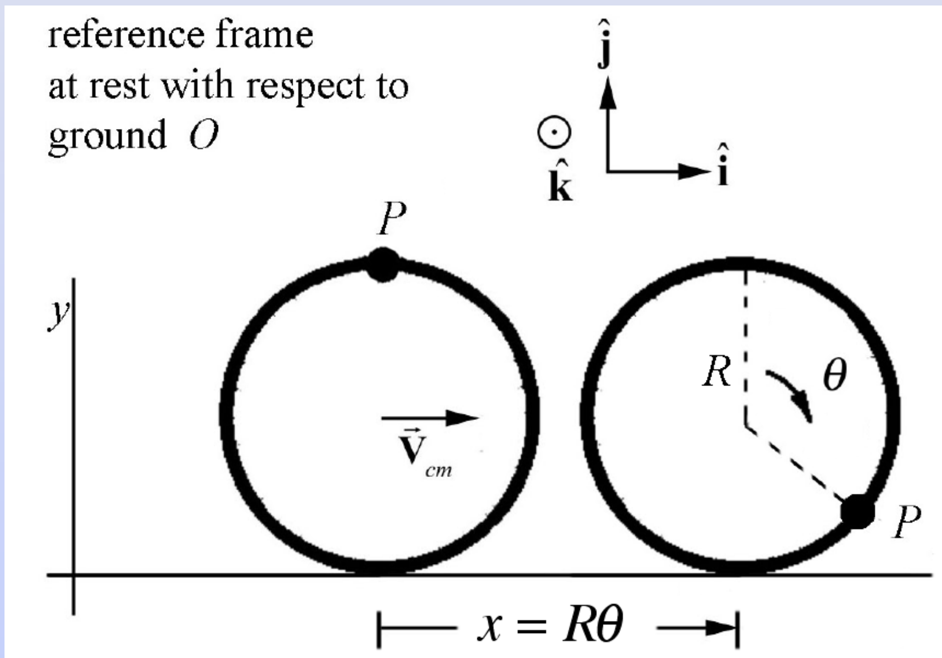
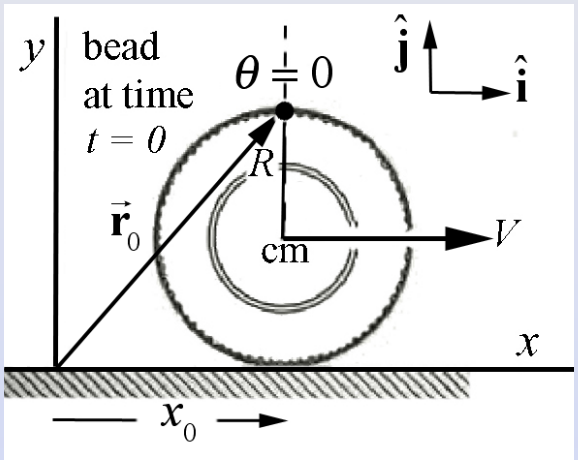
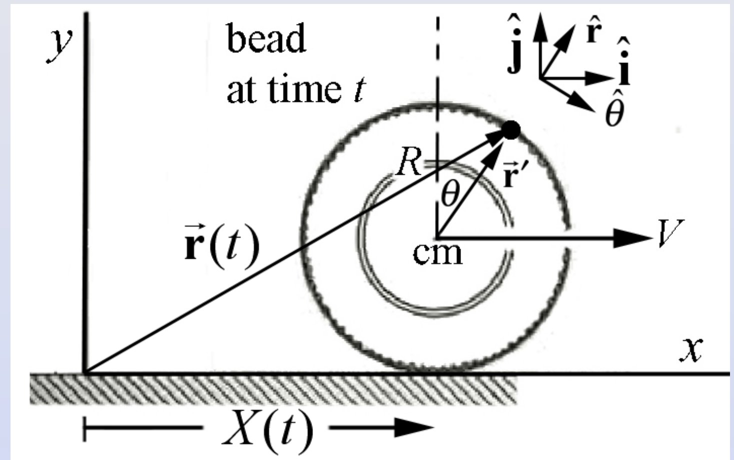


Table Problem: Bicycle Wheel





A bicycle wheel of radius R is rolling without slipping along a horizontal surface. The center of mass of the bicycle in moving with a constant speed V in the positive x-direction. A bead is lodged on the rim of the wheel. Assume that at $t = 0$, the bead is located at the top of the wheel at $x(t = 0) = x_0$ and $y(t = 0) = 2R$. What are the x- and y-components of the position of the bead as a function of time according to an observer fixed to the ground?

Cycloid

$$x(t) = R(t - \sin t) \quad \text{and} \quad y(t) = R(1 - \cos t)$$

curve traced by a point on the rim of the circular wheel as it rolls along a straight line

Courtesy Wikipedia and Wolfram

Angular Momentum for 2-D Rotation and Translation

The angular momentum for a translating and rotating object is given by

$$\vec{\mathbf{L}}_S = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \vec{\mathbf{p}}^{\mathrm{sys}} + \sum_{i=1}^{i=N} \vec{\mathbf{r}}_{\mathrm{cm,i}} \times m_i \vec{\mathbf{v}}_{\mathrm{cm,i}}$$

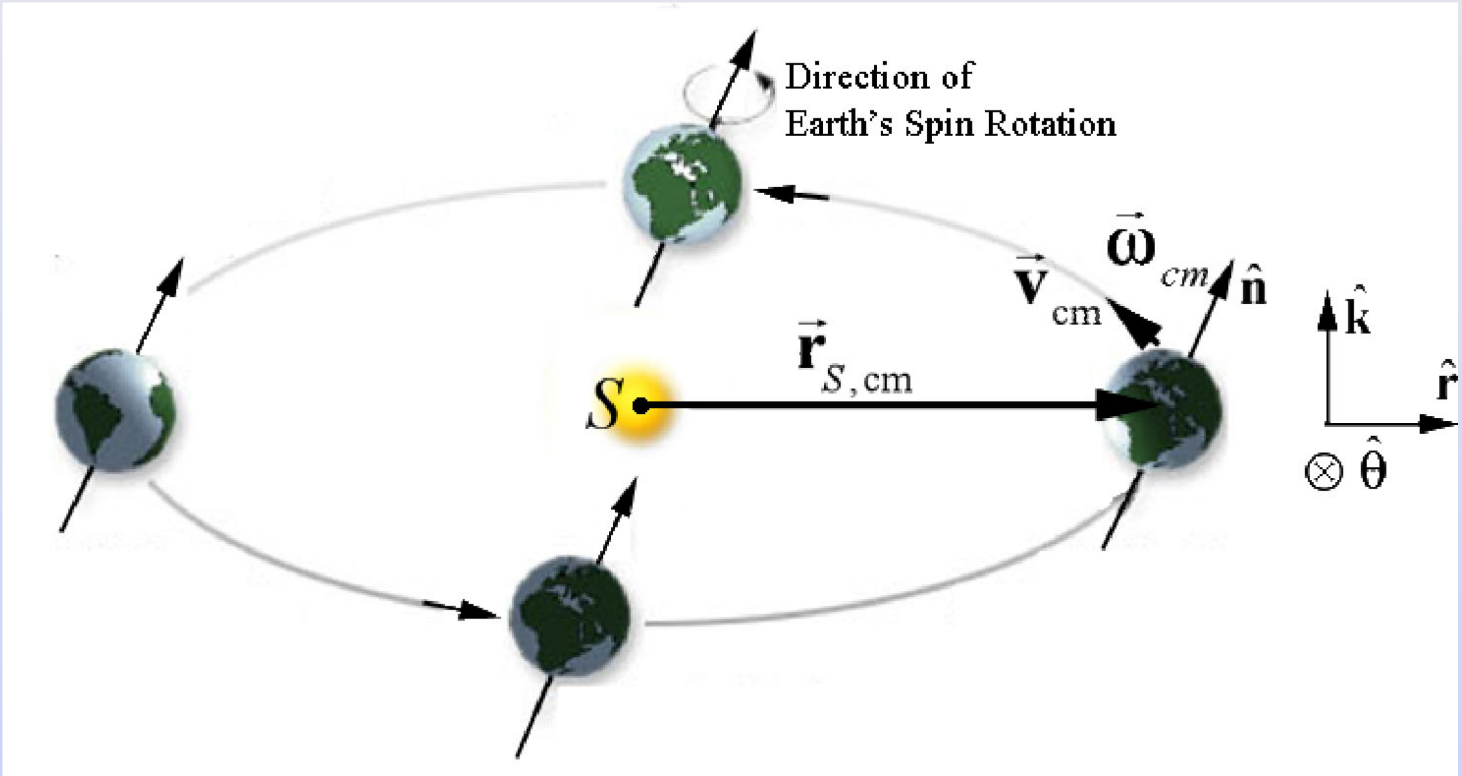
Angular momentum arising from translational of center-of-mass

$$\vec{\mathbf{L}}_{S,\mathrm{cm}} = \vec{\mathbf{R}}_{S,\mathrm{cm}} \times \vec{\mathbf{p}}^{\mathrm{sys}}$$

The second term is the angular momentum arising from rotation about center-of mass,

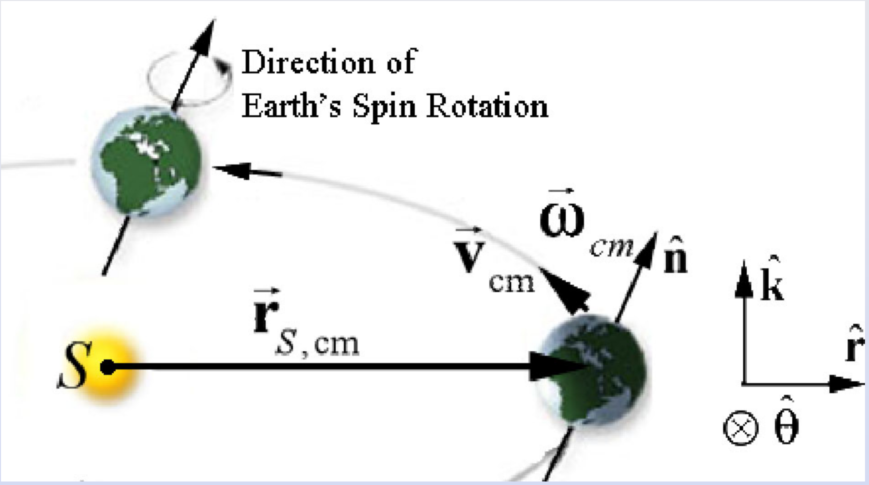
$$\vec{\mathbf{L}}_{\mathrm{cm}} = I_{cm} \vec{\boldsymbol{\omega}}_{cm}$$

Table Problem: Angular Momentum for Earth



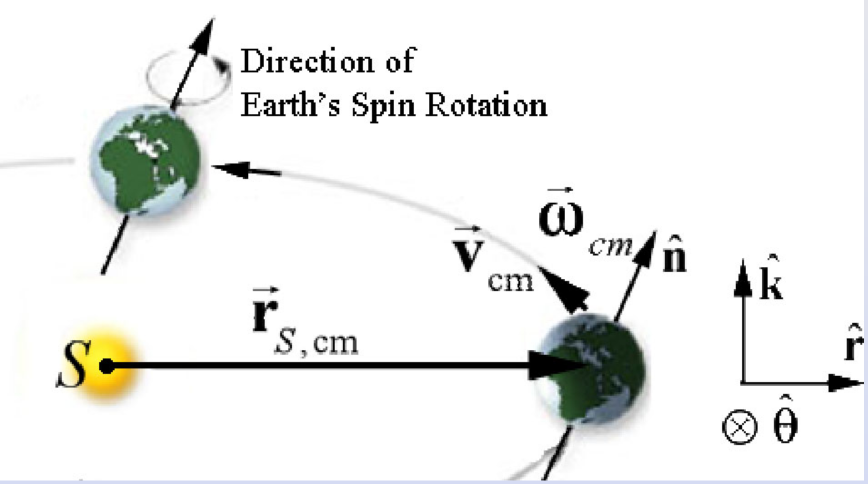
What is the total angular momentum of the Earth as it orbits the Sun? Consider the angular momentum of the center of mass motion of the Earth as well as the angular momentum about its center of mass.

Earth's Orbital Angular Momentum



- Orbital angular momentum about center of sun $\vec{\mathbf{L}}_S^{orbital} = \vec{\mathbf{r}}_{S,cm} \times \vec{\mathbf{p}}^{total} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}}$
- Center of mass velocity and angular velocity $v_{cm} = r_{s,e} \omega_{orbit}$
- Period and angular velocity $\omega_{orbit} = (2\pi / T_{orbit}) = 2.0 \times 10^{-7} \,\mathrm{rad} \cdot \mathrm{s}^{-1}$
- Magnitude $\vec{\mathbf{L}}_S^{orbital} = m_e r_{s,e}{}^2 \omega_{orbit} \hat{\mathbf{k}} = \dfrac{m_e r_{s,e}{}^2 2\pi}{T_{orbit}} \hat{\mathbf{k}}$ $\vec{\mathbf{L}}_S^{orbital} = 2.67 \times 10^{40} \, kg \cdot m^2 \cdot s^{-1} \hat{\mathbf{k}}$

Earth's Spin Angular Momentum

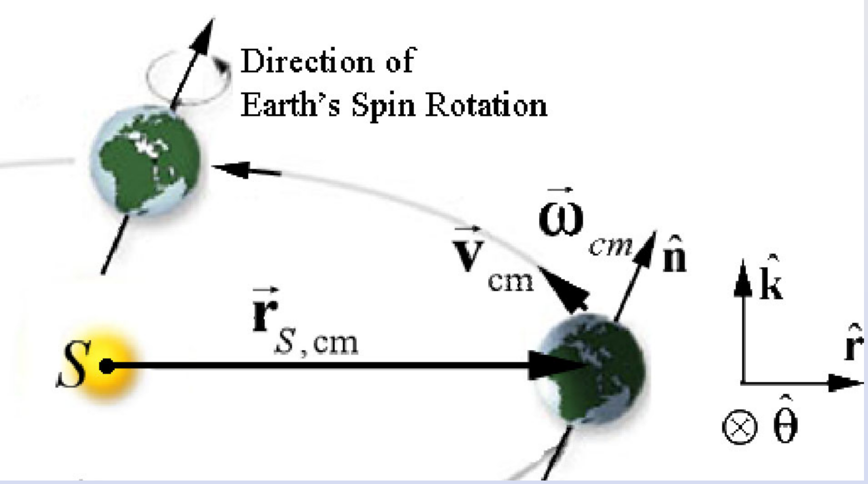


- Spin angular momentum about center of mass of earth $\vec{\mathbf{L}}_{cm}^{spin} = I_{cm}\vec{\boldsymbol{\omega}}_{spin} = \dfrac{2}{5} m_e R_e^{\,2} \omega_{spin} \hat{\mathbf{n}}$

- Period and angular velocity $\omega_{spin} = \dfrac{2\pi}{T_{spin}} = 7.29 \times 10^{-5}\,\text{rad} \cdot \text{s}^{-1}$

- Magnitude $\vec{\mathbf{L}}_{cm}^{spin} = 7.09 \times 10^{33}\, kg \cdot m^2 \cdot s^{-1} \hat{\mathbf{n}}$

Earth's Angular Momentum



For a body undergoing orbital motion like the earth orbiting the sun, the two terms can be thought of as an orbital angular momentum about the center-of-mass of the earth-sun system, denoted by *S*,

$$\vec{\mathbf{L}}_{S,\text{cm}} = \vec{\mathbf{R}}_{S,\text{cm}} \times \vec{\mathbf{p}}^{\text{sys}} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}}$$

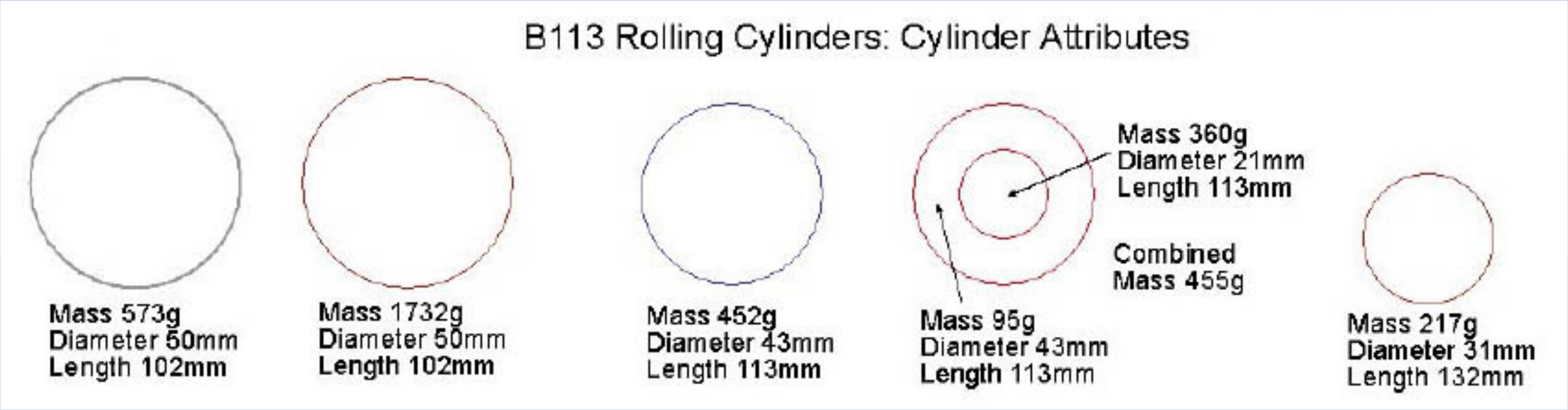
Spin angular momentum about center-of-mass of earth

$$\vec{\mathbf{L}}_{\text{cm}}^{\text{spin}} = I_{\text{cm}} \vec{\boldsymbol{\omega}}_{\text{spin}} = \frac{2}{5} m_e R_e^{\,2} \omega_{\text{spin}} \hat{\mathbf{n}}$$

Total angular momentum about *S*

$$\vec{\mathbf{L}}_S^{\text{total}} = r_{s,e} m_e v_{cm} \hat{\mathbf{k}} + \frac{2}{5} m_e R_e^2 \omega_{\text{spin}} \hat{\mathbf{n}}$$

Demo B 113: Rolling Cylinders



Concept Question: Cylinder Race

Two cylinders of the same size and mass roll down an incline, starting from rest. Cylinder A has most of its mass concentrated at the rim, while cylinder B has most of its mass concentrated at the center. Which reaches the bottom first?

1. A
2. B
3. Both at the same time

Concept Q. Answer : Cylinder Race

Answer 2: Because the moment of inertia of cylinder B is smaller, more of the mechanical energy will go into the translational kinetic energy hence B will have a greater center of mass speed and hence reach the bottom first.

Concept Question: Cylinder race with different masses

Two cylinders of the same size but different masses roll down an incline, starting from rest. Cylinder A has a greater mass. Which reaches the bottom first?

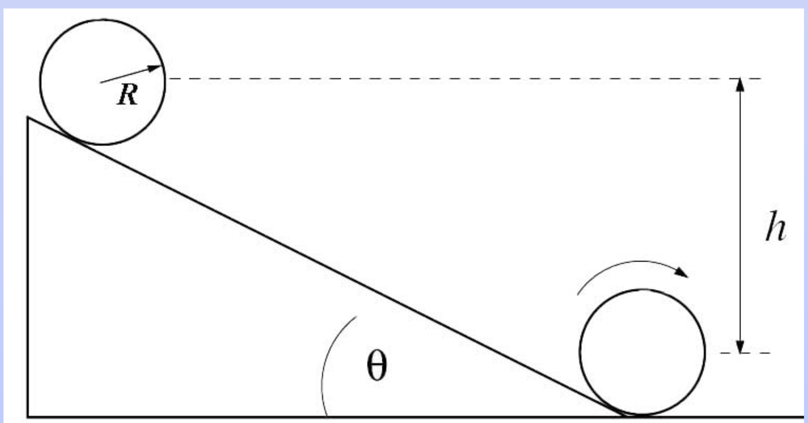
1. A
2. B
3. Both at the same time

Concept Q. Answer: Cylinder race with different masses

Answer 3. The initial mechanical energy is all potential energy and hence proportional to mass. When the cylinders reach the bottom of the incline, both the mechanical energy consists of translational and rotational kinetic energy and both are proportional to mass. So as long as mechanical energy is constant, the final velocity is independent of mass. So both arrive at the bottom at the same time.

Table Problem: Cylinder on inclined plane

A very thin hollow cylinder of outer radius R and mass m with moment of inertia $I_{cm} = M R^2$ about the center of mass starts from rest and moves down an incline tilted at an angle θ from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is μ_s. The cylinder rolls without slipping down the incline. Using energy techniques calculate the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.



Rotational and Translational Motion Dynamics

8.01
W11D2

W11D1 and W11D2 Reading Assignment:
MIT 8.01 Course Notes
Chapter 20 Rigid Body: Translation and Rotational Motion Kinematics for Fixed Axis Rotation
Sections 20.1-20.5
Chapter 21 Rigid Body Dynamics: Rotation and Translation about a Fixed Axis,
Sections 21.1-21.5

Announcements

Sections 1-4 No Class Week 11 Monday

Sunday Tutoring in 26-152 from 1-5 pm

Problem Set 8 due Week 11 Tuesday at 9 pm in box outside 26-152

No Math Review Week 11

Exam 3 Tuesday Nov 26 7:30-9:30 pm
Conflict Exam 3 Wednesday Nov 27 8-10 am, 10-12 noon

Nov 27 Drop Date

Angular Momentum and Torque

1) About any fixed point S

$$\vec{\mathbf{L}}_S = \vec{\mathbf{L}}_S^{orbital} + \vec{\mathbf{L}}_{cm}^{spin} = \vec{\mathbf{r}}_{s,cm} \times m_T \vec{\mathbf{v}}_{cm} + \vec{\mathbf{L}}_{cm}^{spin}$$

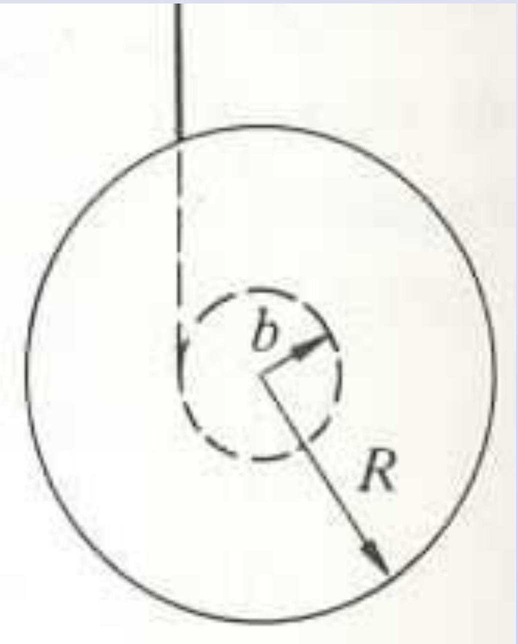
$$\vec{\boldsymbol{\tau}}_S = \sum_i \vec{\boldsymbol{\tau}}_{S,i}^{ext} = \frac{d\vec{\mathbf{L}}_S}{dt}$$

2) Decomposition: $\vec{\boldsymbol{\tau}}_{cm}^{\text{ext}} = \dfrac{d\vec{\mathbf{L}}_{cm}^{spin}}{dt}$

$$\vec{\boldsymbol{\tau}}_{S,cm}^{\text{ext}} = \frac{d\vec{\mathbf{L}}_S^{orbital}}{dt}$$

Worked Example: Descending and Ascending Yo-Yo

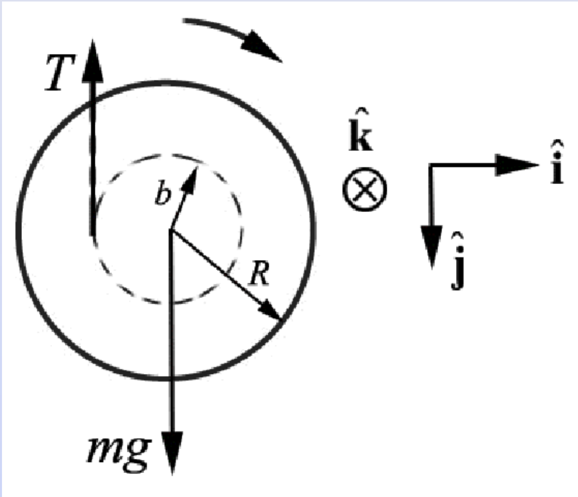
A Yo-Yo of mass m has an axle of radius b and a spool of radius R. It’s moment of inertia about the center of mass can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is released from rest. What is the acceleration of the Yo-Yo as it descends.



Worked Example: Descending and Ascending Yo-Yo

Torque about cm:

$$\vec{\tau}_{\text{cm}} = \vec{\mathbf{r}}_{\text{cm},T} \times \vec{\mathbf{T}} = -b\,\hat{\mathbf{i}} \times -T\,\hat{\mathbf{j}} = bT\,\hat{\mathbf{k}}$$



Torque equation:

$$bT = I\alpha$$

Newton's Second Law: $mg - T = ma_y$

Constraint: $a_y = b\alpha_z$

Tension: $T = Ia_y / b^2$

Acceleration: $$a_y = \frac{mb^2}{(mb^2 + I)}g$$

Demo B107: Descending and Ascending Yo-Yo

$$M_{\text{wheel+axle}} = 435 \text{ g}$$

$$R_{\text{outer}} \cong 6.3 \text{ cm}$$

$$R_{\text{inner}} \cong 4.9 \text{ cm}$$

$$I_{\text{cm}} \cong \frac{1}{2} M \left(R_{\text{outer}}{}^2 + R_{\text{inner}}{}^2 \right)$$

$$= 1.385 \times 10^4 \text{ g} \cdot \text{cm}^2$$

Demonstration:

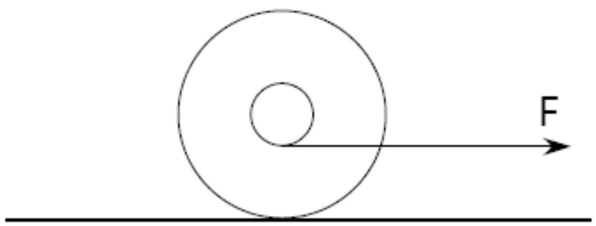
Pulling a Yo-Yo

Mini-Experiment: Spool Yo-Yo

1. Which way does the yo-yo roll when you pull it horizontal?

2. Is there some angle at which you can pull the string in which the yo-yo doesn't roll forward or back?

Concept Question: Pulling a Yo-Yo 1



Two disks are separated by a spindle of smaller diameter. A string is wound around the spindle and pulled gently. In which direction does it roll?

1) To the right, in the direction of F, winding up the string

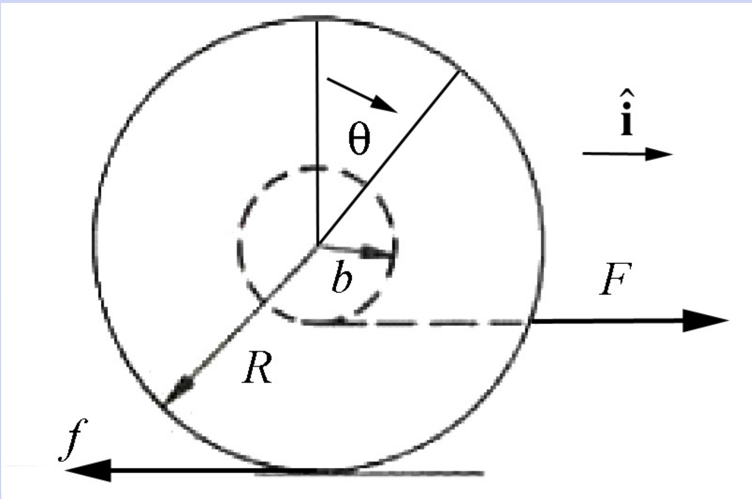
2) To the left, opposite to F, unwinding the string

3) It does not roll, it slides to the right

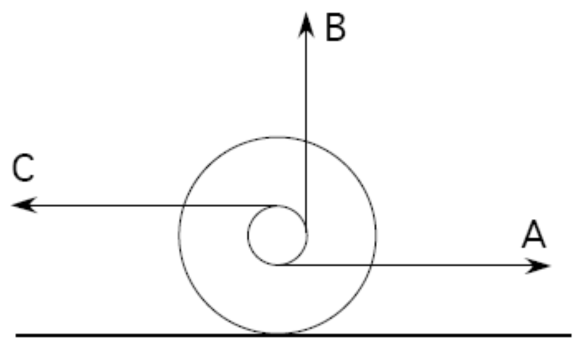
4) It does not roll, it slides to the left

Concept Q. Ans.: Pulling a Yo-Yo 1

Answer 1. For forces below a fixed maximum value, the torque due to the force of friction is larger in magnitude than the torque due to the pulling force. Therefore the cylinder has an angular acceleration pointing into the page (in the clockwise direction) hence the cylinder rolls to the right, in the direction of F, winding up the string.



Concept Question: Pulling a Yo-Yo 2



Two disks are separated by a spindle of smaller diameter. A string is wound around the spindle and pulled gently. At what position of the string does the direction of rotation change?

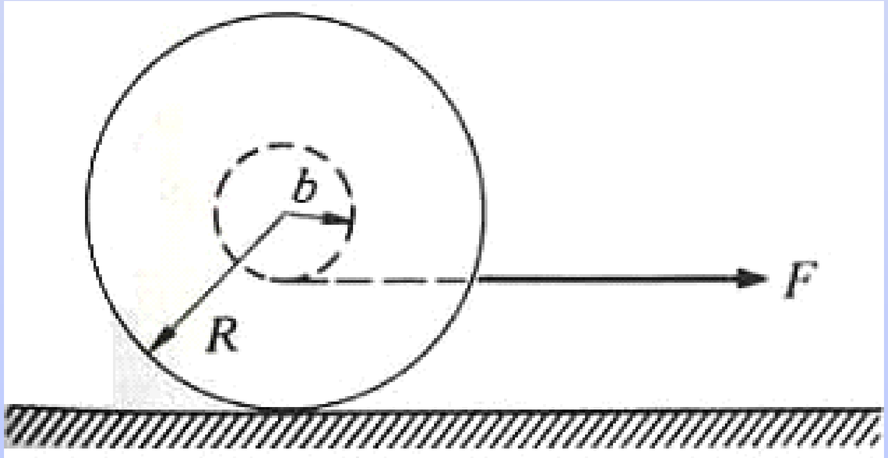
1) At A
2) Somewhere between A and B
3) At B
4) Somewhere between B and C
5) At C
6) It always rolls to the right

Concept Q. Ans.: Pulling a Yo-Yo 2

Answer 2. When the string is pulled up, the only horizontal force is static friction and it points to the left so the yo-yo accelerates to the left. Therefore somewhere between A and B the direction of rotation changes.

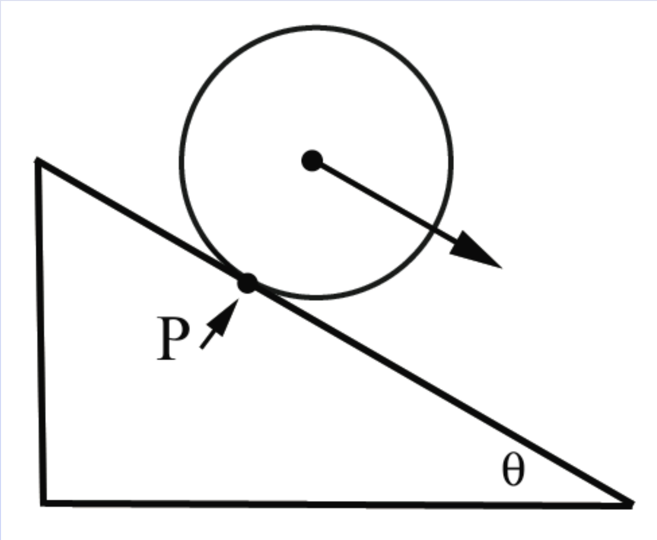
Table Problem: Pulling a Yo-Yo

A Yo-Yo of mass m has an axle of radius b and a spool of radius R. It's moment of inertia about the center of mass can be taken to be $I = (1/2)mR^2$ and the thickness of the string can be neglected. The Yo-Yo is placed upright on a table and the string is pulled with a horizontal force to the right as shown in the figure. The coefficient of static friction between the Yo-Yo and the table is μ_s. What is the maximum magnitude of the pulling force, F, for which the Yo-Yo will roll without slipping?



Concept Question: Cylinder Rolling Down Inclined Plane

A cylinder is rolling without slipping down an inclined plane. The friction at the contact point P is



1. Static and points up the inclined plane.
2. Static and points down the inclined plane.
3. Kinetic and points up the inclined plane.
4. Kinetic and points down the inclined plane.
5. Zero because it is rolling without slipping.

Concept Q. Ans.: Cylinder Rolling Down Inclined Plane

Answer 1. The friction at the contact point P is static and points up the inclined plane. This friction produces a torque about the center of mass that points into the plane of the figure. This torque produces an angular acceleration into the plane, increasing the angular speed as the cylinder rolls down.

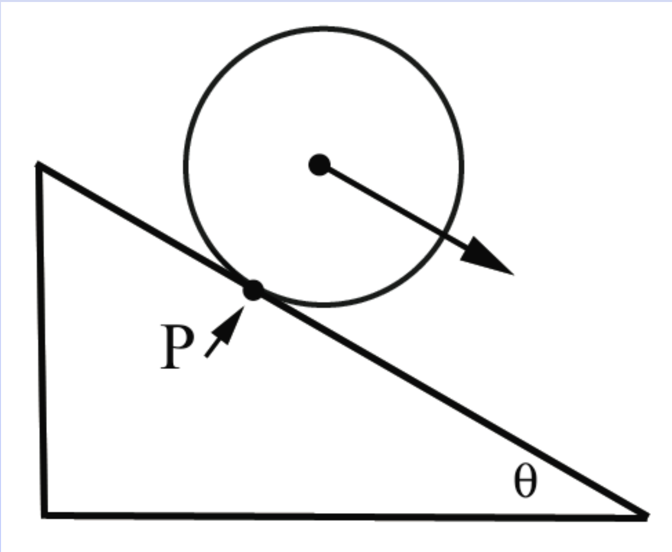
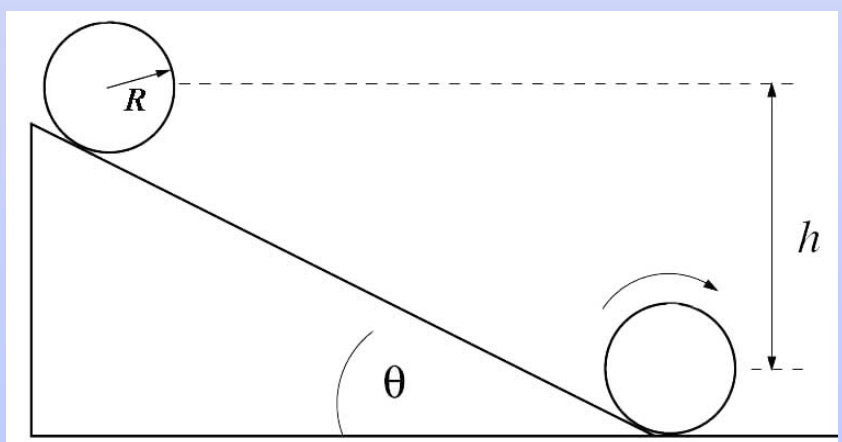
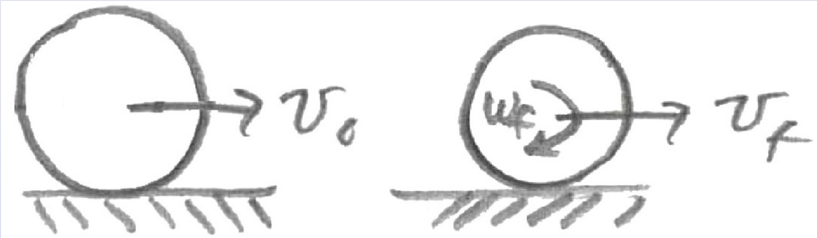


Table Problem: Cylinder on Inclined Plane Torque About Center of Mass

A hollow cylinder of outer radius R and mass m with moment of inertia I_{cm} about the center of mass starts from rest and moves down an incline tilted at an angle q from the horizontal. The center of mass of the cylinder has dropped a vertical distance h when it reaches the bottom of the incline. Let g denote the gravitational constant. The coefficient of static friction between the cylinder and the surface is m_s. The cylinder rolls without slipping down the incline. Using the torque method about the center of mass, calculate the velocity of the center of mass of the cylinder when it reaches the bottom of the incline.



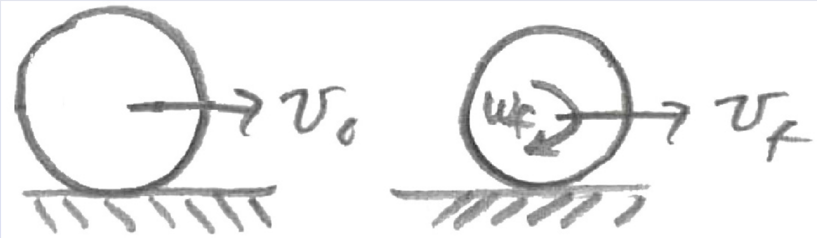
Concept Question: Constants of the Motion



A bowling ball is initially thrown down an alley with an initial speed v_0, and it slides without rolling but due to friction it begins to roll until it rolls without slipping. What quantities are constant before it rolls without slipping?

1. Energy.
2. Angular momentum about the center of mass.
3. Angular momentum about a fixed point on the ground.
4. Three of the above.
5. Two of the above.
6. One of the above.
7. None of the above.

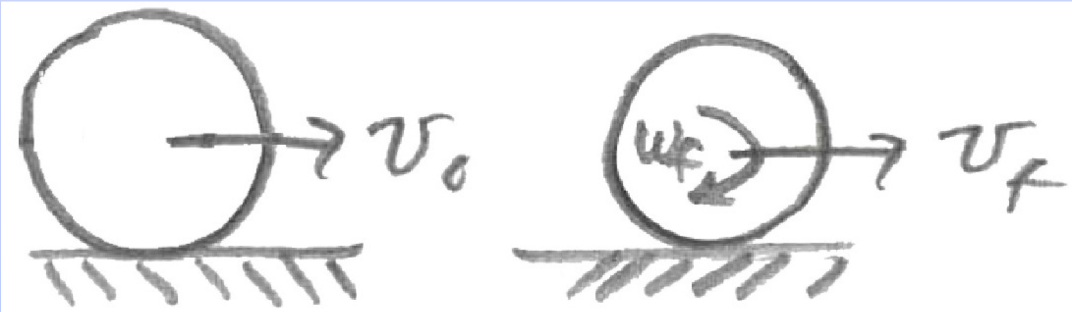
Concept Q. Ans.: Constants of the Motion



Answer 3. Energy is not conserved because there are energy losses due to kinetic friction. Angular momentum about the center of mass is not constant because the friction exerts a torque about the center of mass. Angular momentum about a fixed point on the ground is constant because the sum of the torques about that point is zero. The friction force will always be parallel to the line of contact between the bowling bowl and the surface. So, if we pick any fixed point along the line of contact between the bowling bowl and the surface then the vector from the point to the contact point where friction acts is either parallel or anti-parallel and hence the torque is zero. The torque about a fixed point on the ground due to the gravitational force and the normal forces are in opposite directions, have the same moment arms, and because mg = N have the same magnitudes hence add to zero.

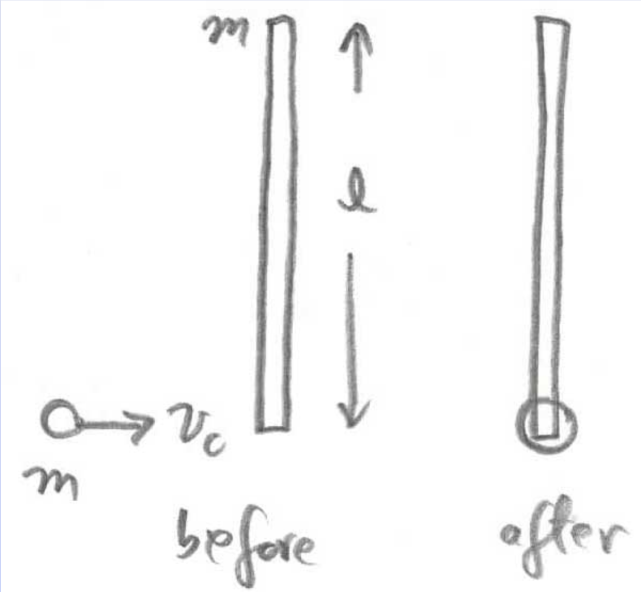
Table Problem: Bowling Ball Conservation of Angular Momentum Method

A bowling ball of mass m and radius R is initially thrown down an alley with an initial speed v_0, and it slides without rolling but due to friction it begins to roll. The moment of inertia of the ball about its center of mass is $I = (2/5)mR^2$. By cleverly choosing a point about which to calculate the angular momentum, use conservation of angular momentum to find is the velocity of the center-of-mass when the wheel rolls without slipping.



Concept Question: Angular Collisions

A long narrow uniform stick lies motionless on ice (assume the ice provides a frictionless surface). The center of mass of the stick is the same as the geometric center (at the midpoint of the stick). A puck (with putty on one side) slides without spinning on the ice toward the stick, hits one end of the stick, and attaches to it.



Which quantities are constant?

1. Angular momentum of puck about center of mass of stick.
2. Momentum of stick and ball.
3. Angular momentum of stick and ball about any point.
4. Mechanical energy of stick and ball.
5. None of the above 1-4.
6. Three of the above 1.4
7. Two of the above 1-4.

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Answer: 7

(2) and (3) are correct. There are no external forces acting on this system so the momentum of the center of mass is constant (1). There are no external torques acting on the system so the angular momentum of the system about any point is constant (3) . However there is a collision force acting on the puck, so the torque about the center of the mass of the stick on the puck is non-zero, hence the angular momentum of puck about center of mass of stick is not constant. The mechanical energy is not constant because the collision between the puck and stick is inelastic.

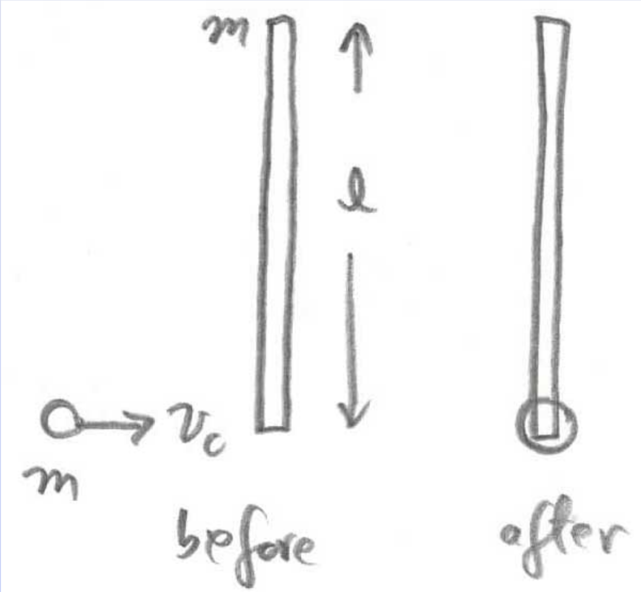

Table Problem: Angular Collision

A long narrow uniform stick of length *l* and mass *m* **lies motionless on a frictionless**). The moment of inertia of the stick about its center of mass is I_{cm}. A puck (with putty on one side) has the same mass *m* as the stick. The puck slides without spinning on the ice with a speed of v_0 toward the stick, hits one end of the stick, and attaches to it. (You may assume that the radius of the puck is much less than the length of the stick so that the moment of inertia of the puck about its center of mass is negligible compared to I_{cm}.) What is the angular velocity of the stick plus puck after the collision?