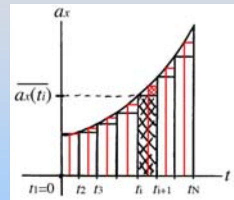
Non-Uniform Acceleration, Vectors, Kinematics in Two-Dimensions

8.01

W02D2

Velocity as the Integral of the Acceleration

the area under the graph of the *x*-component of the acceleration vs. time is the change in velocity



$$\int_{t'=0}^{t'=t} a_x(t')dt' \equiv \lim_{\Delta t_i \to 0} \sum_{i=1}^{i=N} a_x(t_i)\Delta t_i = Area(a_x, t)$$

$$\int_{t'=0}^{t'=t} a_x(t')dt' = \int_{t'=0}^{t'=t} \frac{dv_x}{dt} dt' = \int_{v'_x = v_x(t=0)}^{v'_x = v_x(t)} dv'_x = v_x(t) - v_{x,0}$$

Non-Uniform Acceleration

Position as the Integral of Velocity

Area under the graph of *x*-component of the velocity vs. time is the *x*-component of the displacement

$$v_x(t) \equiv \frac{dx}{dt}$$

$$\int_{t'=0}^{t'=t} v_x(t')dt' = x(t) - x_0$$

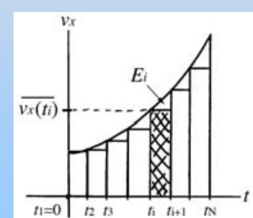


Table Problem: Time Dependent Acceleration

Consider an object released at time $t = 0$ with an initial x-component of velocity $v_{x,0}$, located at position x_0 ,and accelerating according to

$$a_x(t) = b_0 - b_1 t$$

Find the velocity and position as a function of time.

Concept Question

Consider an object released at time $t = 0$ with an initial x-component of velocity $v_{x,0} = 0$,and accelerating according to

$$a_x \equiv \frac{dv_x}{dt} = c_0 - c_1 v_x$$

After a very long time, the x-component of the velocity is

1. Zero
2. c_0-c_1
3. c_0/c_1
4. c_0+c_1
5. Not sure

Table Problem: Time Dependent Acceleration

Velocity:

$$v_x(t) = v_{x,0} + \int_{t'=0}^{t'=t} a_x(t')dt'$$

$$= v_{x,0} + \int_{t'=0}^{t'=t} (b_0 - b_1 t')dt' = v_{x,0} + b_0 t'\Big|_{t'=0}^{t'=t} - \frac{1}{2} b_1 t'^2\Big|_{t'=0}^{t'=t}$$

$$= v_{x,0} + b_0 t - \frac{1}{2} b_1 t^2$$

Position:

$$x(t) = x_0 + \int_{t'=0}^{t'=t} v_x(t')dt' =$$

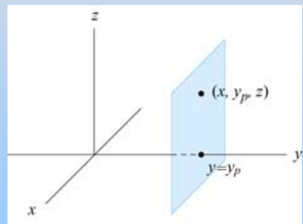
$$= x_0 + \int_{t'=0}^{t'=t} (v_{x,0} + b_0 t' - \frac{1}{2} b_1 t'^2)dt' = v_{x,0} t + \frac{1}{2} b_0 t^2 - \frac{1}{6} b_1 t^3$$

Vectors

Coordinate System

Coordinate system: used to describe the position of a point in space and consists of

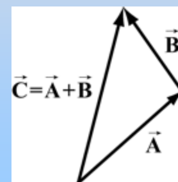
1. An origin as the reference point
2. A set of coordinate axes with scales and labels
3. Choice of positive direction for each axis
4. Choice of unit vectors at each point in space

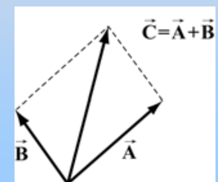


Cartesian Coordinate System

Vector Addition

Let $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ be two vectors. Define a new vector $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$,the "vector addition" of $\vec{\mathbf{A}}$ and $\vec{\mathbf{B}}$ by the geometric construction shown in either figure

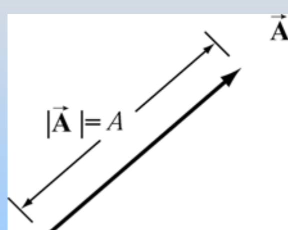




Vector

A vector is a quantity that has both direction and magnitude. Let a vector be denoted by the symbol $\vec{\mathbf{A}}$

The magnitude of $\vec{\mathbf{A}}$ is denoted by $|\vec{\mathbf{A}}| \equiv A$



Summary: Vector Properties

Addition of Vectors

1. Commutativity $\vec{\mathbf{A}} + \vec{\mathbf{B}} = \vec{\mathbf{B}} + \vec{\mathbf{A}}$
2. Associativity $(\vec{\mathbf{A}} + \vec{\mathbf{B}}) + \vec{\mathbf{C}} = \vec{\mathbf{A}} + (\vec{\mathbf{B}} + \vec{\mathbf{C}})$
3. Identity Element for Vector Addition $\vec{\mathbf{0}}$ such that $\vec{\mathbf{A}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} + \vec{\mathbf{A}} = \vec{\mathbf{A}}$
4. Inverse Element for Vector Addition $-\vec{\mathbf{A}}$ such that $\vec{\mathbf{A}} + \left(-\vec{\mathbf{A}}\right) = \vec{\mathbf{0}}$

Scalar Multiplication of Vectors

1. Associative Law for Scalar Multiplication $b(c\vec{\mathbf{A}}) = (bc)\vec{\mathbf{A}} = (cb\,\vec{\mathbf{A}}) = c(b\vec{\mathbf{A}})$
2. Distributive Law for Vector Addition $c(\vec{\mathbf{A}} + \vec{\mathbf{B}}) = c\,\vec{\mathbf{A}} + c\,\vec{\mathbf{B}}$
3. Distributive Law for Scalar Addition $(b + c)\,\vec{\mathbf{A}} = b\,\vec{\mathbf{A}} + c\,\vec{\mathbf{A}}$
4. Identity Element for Scalar Multiplication: number 1 such that $1\,\vec{\mathbf{A}} = \vec{\mathbf{A}}$

Application of Vectors

(1) Vectors can exist at any point *P* in space.

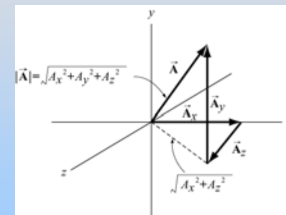
(2) Vectors have direction and magnitude.

(3) Vector Equality: Any two vectors that have the same direction and magnitude are equal no matter where in space they are located.

Unit Vectors and Components

The idea of multiplication by real numbers allows us to define a set of unit vectors at each point in space $(\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}})$

with $|\hat{\mathbf{i}}|=1, |\hat{\mathbf{j}}|=1, |\hat{\mathbf{k}}|=1$



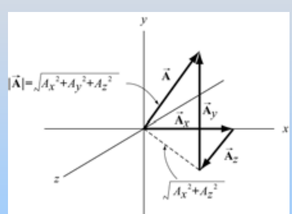
Components:

$$\vec{\mathbf{A}} = (A_x, A_y, A_z)$$

$$\vec{\mathbf{A}}_x = A_x\, \hat{\mathbf{i}},\ \vec{\mathbf{A}}_y = A_y\, \hat{\mathbf{j}},\quad \vec{\mathbf{A}}_z = A_z\, \hat{\mathbf{k}} \qquad \vec{\mathbf{A}} = A_x\, \hat{\mathbf{i}} + A_y\, \hat{\mathbf{j}} + A_z\, \hat{\mathbf{k}}$$

Vector Decomposition

Choose a coordinate system with an origin and axes. We can decompose a vector into component vectors along each coordinate axis, for example along the x,y, and z-axes of a Cartesian coordinate system. A vector at *P* can be decomposed into the vector sum,



$$\vec{\mathbf{A}} = \vec{\mathbf{A}}_x + \vec{\mathbf{A}}_y + \vec{\mathbf{A}}_z$$

Vector Decomposition in Two Dimensions

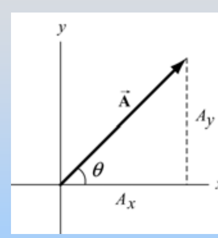
Consider a vector

$$\vec{\mathbf{A}} = (A_x, A_y, 0)$$

x- and y components:

$$A_x = A\cos(\theta), \quad A_y = A\sin(\theta)$$

Magnitude: $A = \sqrt{A_x^2 + A_y^2}$



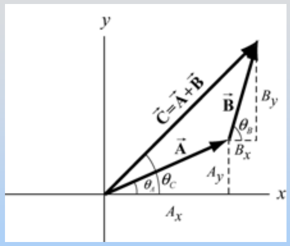
Direction: $\frac{A_y}{A_x} = \frac{A\sin(\theta)}{A\cos(\theta)} = \tan(\theta)$

$$\theta = \tan^{-1}(A_y / A_x)$$

Vector Addition

$$\vec{\mathbf{A}} = A\cos(\theta_A)\ \hat{\mathbf{i}} + A\sin(\theta_A)\ \hat{\mathbf{j}}$$

$$\vec{\mathbf{B}} = B\cos(\theta_B)\ \hat{\mathbf{i}} + B\sin(\theta_B)\ \hat{\mathbf{j}}$$



Vector Sum: $\vec{\mathbf{C}} = \vec{\mathbf{A}} + \vec{\mathbf{B}}$

Components

$$C_x = A_x + B_x, \quad C_y = A_y + B_y$$

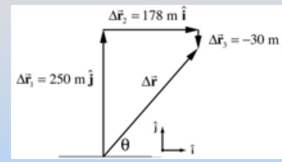
$$C_x = C\cos(\theta_C) = A\cos(\theta_A) + B\cos(\theta_B)$$

$$C_y = C\sin(\theta_C) = A\sin(\theta_A) + B\sin(\theta_B)$$

$$\vec{\mathbf{C}} = (A_x + B_x)\ \hat{\mathbf{i}} + (A_y + B_y)\ \hat{\mathbf{j}} = C\cos(\theta_C)\ \hat{\mathbf{i}} + C\sin(\theta_C)\ \hat{\mathbf{j}}$$

Table Problem: Solution

Total displacement:



$$\begin{aligned}\Delta\vec{\mathbf{r}} &= \Delta\vec{\mathbf{r}}_1 + \Delta\vec{\mathbf{r}}_2 + \Delta\vec{\mathbf{r}}_3 \\ &= 250\ \text{m}\ \hat{\mathbf{j}} + 178\ \text{m}\ \hat{\mathbf{i}} + -30\ \text{m}\ \hat{\mathbf{j}} \\ &= 178\ \text{m}\ \hat{\mathbf{i}} + 220\ \text{m}\ \hat{\mathbf{j}}\end{aligned}$$

Magnitude:

$$|\Delta\vec{\mathbf{r}}| = ((178\ \text{m})^2 + (220\ \text{m})^2)^{1/2} = 283\ \text{m}$$

Direction:

$$\begin{aligned}\theta &= \tan^{-1}((\vec{\Delta} r)_y / (\vec{\Delta} r)_x) \\ &= \tan^{-1}(220\ \text{m} / 178\ \text{m}) = 51.0^o\end{aligned}$$

Table Problem: Displacement Vector

At 2 am one morning a person runs 250 m along the infinite corridor at MIT from Mass Ave to the end of Building 8, turns right at the end of the corridor and runs 178 m to the end of Building 2, and then turns right and runs 30 m down the hall.

What is the direction and magnitude of the straight line between start and finish?

Vector Description of Motion

- Position $\vec{\mathbf{r}}(t) = x(t)\hat{\mathbf{i}} + y(t)\hat{\mathbf{j}}$
- Displacement $\Delta\vec{\mathbf{r}}(t) = \Delta x(t)\,\hat{\mathbf{i}} + \Delta y(t)\,\hat{\mathbf{j}}$
- Velocity $\vec{\mathbf{v}}(t) = \dfrac{dx(t)}{dt}\hat{\mathbf{i}} + \dfrac{dy(t)}{dt}\hat{\mathbf{j}} \equiv v_x(t)\hat{\mathbf{i}} + v_y(t)\hat{\mathbf{j}}$
- Acceleration $\vec{\mathbf{a}}(t) = \dfrac{dv_x(t)}{dt}\hat{\mathbf{i}} + \dfrac{dv_y(t)}{dt}\hat{\mathbf{j}} \equiv a_x(t)\hat{\mathbf{i}} + a_y(t)\hat{\mathbf{j}}$

Constant Acceleration

- Components of Velocity:

$$v_x = v_{0,x} + a_x t \ , \qquad v_y = v_{0,y} + a_y t$$

- Components of Position:

$$x = x_0 + v_{x0} t + \frac{1}{2} a_x t^2, \qquad y = y_0 + v_{y0} t + \frac{1}{2} a_y t^2$$

- Eliminating t:

$$2a_x (x - x_0) = v_x^2 - v_{x,0}^2$$

Projectile Motion

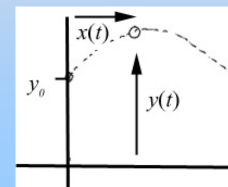
A projectile is fired from a height y_0 with an initial speed v_0 at an angle θ above the horizontal. Ignore air resistance

Gravitational Force Law

$$\vec{\mathbf{F}}_{\text{grav}} = -m_{\text{grav}} g \,\hat{\mathbf{j}} \quad \Rightarrow \quad F_{\text{grav},y} = -m_{\text{grav}} g$$

Newton's Second Law

$$\vec{\mathbf{F}}^{\text{total}} = m_{\text{in}} \vec{\mathbf{a}} \quad \Rightarrow \quad \begin{cases} F_x^{\text{total}} = m_{\text{in}} a_x \\ F_y^{\text{total}} = m_{\text{in}} a_y \end{cases}$$



Two-Dimensional Motion

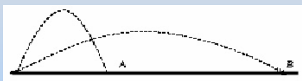
Equations of Motion

- y-component: $-m_{\text{grav}} g = m_{\text{in}} a_y$
- x-component: $0 = m_{\text{in}} a_x$
- Principle of Equivalence: $m_{\text{grav}} = m_{\text{in}}$
- Components of acceleration:

$$a_x = 0, \ a_y = -g \qquad g = 9.8 \text{ m/s}^2$$

Concept Q.: 2-dim kinematics

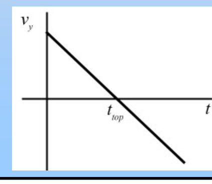
A person simultaneously throws two objects in the air. The objects leave the person's hands at different angles and travel along the parabolic trajectories indicated by A and B in the figure below. Which of the following statements best describes the motion of the two objects?

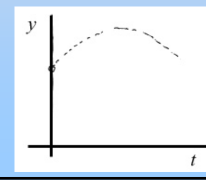


1. The object moving along the trajectory A hits the ground before the object moving along the trajectory B.
2. The object moving along the higher trajectory A hits the ground after the object moving along the lower trajectory B.
3. Both objects hit the ground at the same time.
4. There is not enough information is specified in order to determine which object hits the ground first.

Kinematic Equations *y*-components :

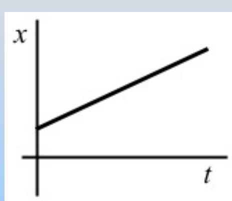
- Acceleration: $a_y = -g$
- Velocity: $v_y(t) = v_{y,0} - gt$
- Position: $y(t) = y_0 + v_{y,0}t - \frac{1}{2}gt^2$





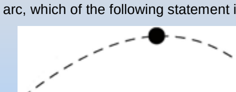
Kinematic Equations -- *x*-components :

- Acceleration : $a_x = 0$
- Velocity : $v_x(t) = v_{x,0}$
- Position : $x(t) = x_0 + v_{x,0}t$



Concept Question: 2-dim kinematics

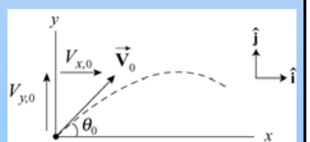
Consider the path of a ball moving along a path through the air under the action of the gravitational force. You may neglect the effects of air friction. As it reaches the highest point in its arc, which of the following statement is true?



1) The magnitudes of the velocity and acceleration are zero.
2) The magnitude of the velocity is at a minimum but not equal to zero.
3) The magnitude of the velocity is equal to zero, and the magnitude of the acceleration is constant and not equal to zero.
4) The magnitude of the velocity is at a minimum but not equal to zero and the magnitude of the acceleration is zero.
5) Neither the magnitudes of acceleration or velocity has yet attained its minimum value.

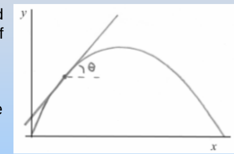
Initial Conditions

- Initial position: $\vec{\mathbf{r}}_0 = x_0\hat{\mathbf{i}} + y_0\hat{\mathbf{j}}$
- Initial velocity: $\vec{\mathbf{v}}_0(t) = v_{x,0}\hat{\mathbf{i}} + v_{y,0}\hat{\mathbf{j}}$
- Velocity components: $v_{x,0} = v_0 \cos\theta_0, \quad v_{y,0} = v_0 \sin\theta_0$
- Initial speed: $v_0 = |\vec{\mathbf{v}}_0| = \left(v_{x,0}^2 + v_{y,0}^2\right)^{1/2}$
- Direction: $\theta_0 = \tan^{-1}\left(\dfrac{v_{y,0}}{v_{x,0}}\right)$



Concept Q.: 2-dim kinematics

An object moves along a parabolic orbit under the influence of gravitation. At each point along the orbit,



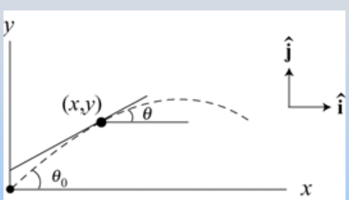
1. the magnitude of the velocity can be determined from the slope of the tangent line to the graph of y vs. x but not the direction
2. the magnitude and direction of the velocity can be determined from the slope of the tangent line to the graph of y vs. x
3. the magnitude and direction of the velocity cannot be determined from the slope of the tangent line to the graph of y vs. x
4. the direction of the velocity can be determined from the slope of the tangent line to the graph of y vs. x but not the magnitude.

Orbit equation

The slope of the curve $y(t)$ vs. $x(t)$ at any point determines the direction of the velocity

$$\theta = \tan^{-1}\left(\frac{dy}{dx}\right)$$

$$(x_0, y_0) = (0,0)$$

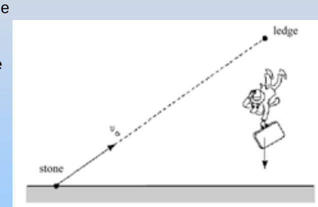


$$x(t) = v_{x,0}t \quad \Rightarrow \quad t = \frac{x(t)}{v_{x,0}}$$

$$y(t) = v_{y,0}t - \frac{1}{2}gt^2 = \frac{v_{y,0}}{v_{x,0}}x(t) - \frac{1}{2}\frac{g}{v_{x,0}^2}x(t)^2$$

Concept Q.: 2-dim kinematics

Consider the situation depicted here. A stone is accurately aimed at a person hanging from the ledge of a building. The target is well within the stone's range, but the instant the stone is thrown, the person lets go and drops to the ground. The stone moves with a speed v_0, just as it is released. What happens?



The stone

1. hits the person, regardless of the value of v_0;
2. hits the person only if v_0 is large enough;
3. misses the person.

Table Problem: Stuffed Animal and the Gun

A stuffed animal is suspended at a height *h* above the ground. A physics demo instructor has set up a projectile gun a horizontal distance *d* away from the stuffed animal. The projectile is initially a height *s* above the ground. The demo instructor fires the projectile with an initial velocity of magnitude v_0 just as the stuffed animal is released. Find the angle the projectile gun must be aimed in order for the projectile to strike the stuffed animal. Ignore air resistance.

Demo:

Stuffed Animal and Gun