Static Equilibrium and Torque

8.01

W04D2

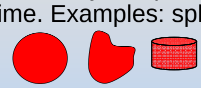
Static Equilibrium for Forces

(1) The sum of the forces acting on a body at rest is zero

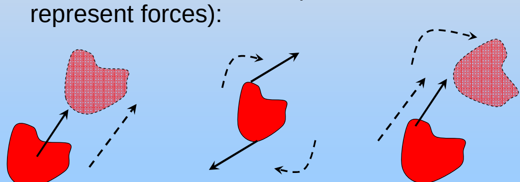
$$\vec{\mathbf{F}}_{\text{total}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \ldots = \vec{\mathbf{0}}$$

Rigid Bodies

- Rigid body: An extended object in which the distance between any two points in the object is constant in time. Examples: sphere, disk …



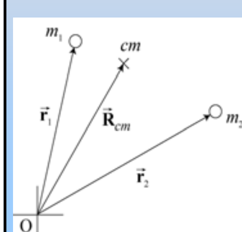
- Effect of external forces (the solid arrows represent forces):



Center of Mass *(C.M.)*

A *special* point in a rigid body:

- The body can be balanced by pivoting it on the *c.m.*
- The gravitational force between the Earth and the body is exerted at the *c.m.* (take it as an empirical statement now).



For a rigid body with two point particles:

$$\vec{\mathbf{R}}_{cm} = \frac{m_1\vec{\mathbf{r}}_1 + m_2\vec{\mathbf{r}}_2}{m_1 + m_2}$$

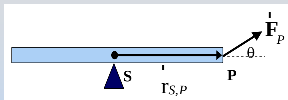
For a rigid body with n- point particles:

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{\sum_{i=1}^{i=N} m_i\vec{\mathbf{r}}_i}{\sum_{i=1}^{i=N} m_i}$$

Torque: Magnitude and Direction

Magnitude of torque about a point S:

$$\tau_S = rF_\perp = rF\sin\theta$$

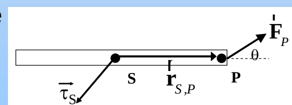


where F is the magnitude of the force $\vec{F}_P$.

Direction of torque:

Perpendicular to the plane formed by $\vec{F}_P$ and $\vec{r}_{S,P}$.

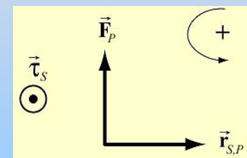
Determined by the ***Right-Hand-rule.***



Sign Convention

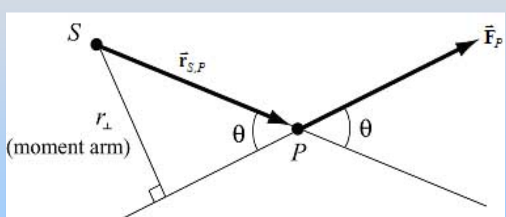
Counterclockwise positive

(Right-Hand Rule)



Line of Action of the Force

- Moment Arm: $r_\perp = r\sin\theta$

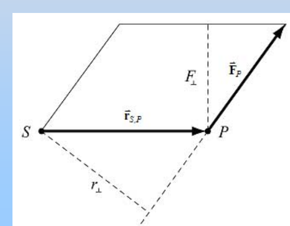


- Torque: $\tau_S = rF_\perp = rF\sin\theta = r_\perp F$

Geometric Interpretation of Torque

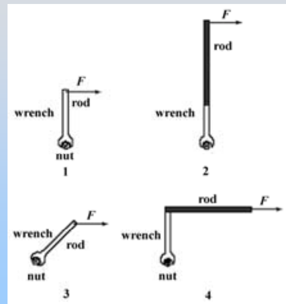
The magnitude of the torque is equal to the area of a parallelogram:

$$A = \tau_S = r_\perp F = rF_\perp$$



Concept Question

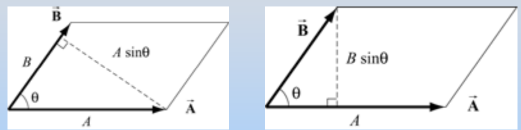
You are using a wrench to tighten a nut. Which of the arrangements shown is most effective in tightening the nut?



Cross Product

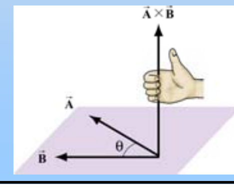
Magnitude:

$$\left|\vec{\mathbf{A}}\times\vec{\mathbf{B}}\right| = AB\sin\theta = A(B\sin\theta) = (A\sin\theta)B \quad (0\le\theta\le\pi)$$



area of the parallelogram

Direction: determined by the Right-Hand-Rule



Properties of Cross Products

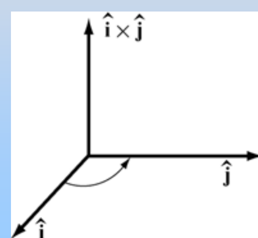
$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = -\vec{\mathbf{B}}\times\vec{\mathbf{A}}$$

$$c(\vec{\mathbf{A}}\times\vec{\mathbf{B}}) = \vec{\mathbf{A}}\times c\vec{\mathbf{B}} = c\vec{\mathbf{A}}\times\vec{\mathbf{B}}$$

$$(\vec{\mathbf{A}}+\vec{\mathbf{B}})\times\vec{\mathbf{C}} = \vec{\mathbf{A}}\times\vec{\mathbf{C}} + \vec{\mathbf{B}}\times\vec{\mathbf{C}}$$

Cross Product of Unit Vectors

- Unit vectors in Cartesian coordinates



$$\left|\hat{\mathbf{i}}\times\hat{\mathbf{j}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(\pi/2) = 1$$

$$\left|\hat{\mathbf{i}}\times\hat{\mathbf{i}}\right| = |\hat{\mathbf{i}}||\hat{\mathbf{j}}|\sin(0) = 0$$

$$\hat{\mathbf{i}}\times\hat{\mathbf{j}} = \hat{\mathbf{k}} \qquad \hat{\mathbf{i}}\times\hat{\mathbf{i}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{j}}\times\hat{\mathbf{k}} = \hat{\mathbf{i}} \qquad \hat{\mathbf{j}}\times\hat{\mathbf{j}} = \vec{\mathbf{0}}$$

$$\hat{\mathbf{k}}\times\hat{\mathbf{i}} = \hat{\mathbf{j}} \qquad \hat{\mathbf{k}}\times\hat{\mathbf{k}} = \vec{\mathbf{0}}$$

Components of Cross Product

$$\vec{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}} + A_z\hat{\mathbf{k}}, \quad \vec{\mathbf{B}} = B_x\hat{\mathbf{i}} + B_y\hat{\mathbf{j}} + B_z\hat{\mathbf{k}}$$

$$\vec{\mathbf{A}}\times\vec{\mathbf{B}} = (A_yB_z - A_zB_y)\hat{\mathbf{i}} + (A_zB_x - A_xB_z)\hat{\mathbf{j}} + (A_xB_y - A_yB_x)\hat{\mathbf{k}}$$

$$= \begin{vmatrix} \hat{\mathbf{i}} & \hat{\mathbf{j}} & \hat{\mathbf{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Concept Question: Cross Product

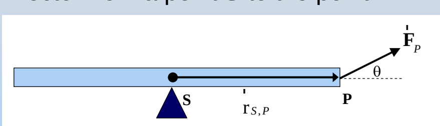
Consider two vectors $\vec{\mathbf{r}} = x\hat{\mathbf{i}}$ with x >0 and $\vec{\mathbf{F}} = F_x\hat{\mathbf{i}} + F_z\hat{\mathbf{k}}$ with $F_x > 0$ and $F_z > 0$. The cross product $\vec{\mathbf{r}}\times\vec{\mathbf{F}}$ points in the

1) + x-direction
2) -x-direction
3) +y-direction
4) -y-direction
5) +z-direction
6) -z-direction
7) None of the above directions

Torque as a vector

$\vec{\mathbf{F}}_P$: Force exerted at a point P on a rigid body.

$\vec{\mathbf{r}}_{S,P}$: Vector from a point *S* to the point *P.*



Torque about point S due to the force exerted at point P:

$$\vec{\tau}_S = \vec{\mathbf{r}}_{S,P}\times\vec{\mathbf{F}}_P$$

Conditions for Static Equilibrium

(1) Translational equilibrium: the sum of the forces acting on the rigid body is zero.

$$\vec{\mathbf{F}}_{\text{total}} = \vec{\mathbf{F}}_1 + \vec{\mathbf{F}}_2 + \ldots = \vec{\mathbf{0}}$$

(2) Rotational Equilibrium: the vector sum of the torques about any point S in a rigid body is zero.

$$\vec{\tau}_{\mathbf{S}}^{\,\mathbf{total}} = \vec{\tau}_{S,1} + \vec{\tau}_{S,2} + \ldots = \vec{\mathbf{0}}$$

Problem Solving Strategy

Force:

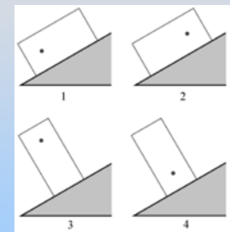
1. Identify System and draw all forces and where they act on Free Body Force Diagram
2. Write down equations for static equilibrium of the forces: sum of forces is zero

Torque:

1. Choose point to analyze the torque about.
2. Choose sign convention for torque
3. Calculate torque about that point for each force. (Note sign of torque.)
4. Write down equation corresponding to condition for static equilibrium: sum of torques is zero

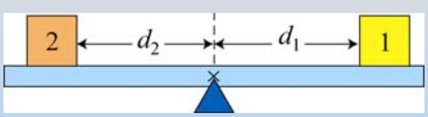
Concept Question: Tipping

A box, with its center-of-mass off-center as indicated by the dot, is placed on an inclined plane. In which of the four orientations shown, if any, does the box tip over?



Lever Law

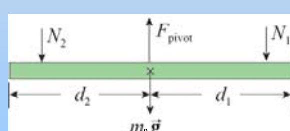
Pivoted Lever at Center of Mass in Equilibrium



$$N_1 = m_1 g,\ N_2 = m_2 g$$

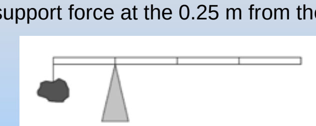
$$d_1 N_1 = d_2 N_2$$

$$d_1 m_1 = d_2 m_2$$



Concept Question

A 1 kg rock is suspended by a massless string from one end of a 1 m measuring stick. What is the mass of the measuring stick if it is balanced by a support force at the 0.25 m from the left end?



1. 0.25 kg.
2. 0.5 kg.
3. 1.0 kg.
4. 2.0 kg.
5. 4.0 kg.
6. Impossible to determine.

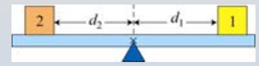
Worked Example 1

Suppose a beam of length s = 1.0 m and mass m = 2.0 kg is balanced on a pivot point that is placed directly beneath the center of the beam. Suppose a mass m_1 = 0.3 kg is placed a distance d_1 = 0.4 m to the right of the pivot point. A second mass m_2 = 0.6 kg is placed a distance d_2 to the left of the pivot point to keep the beam static.

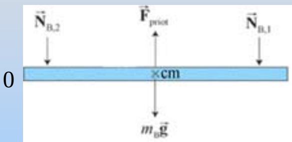
(1) What is the force that the pivot exerts on the beam?

(2) What is the distance d_2 that maintains static equilibrium?

Worked Example: Solution



Apply Newton's 2nd law to each body:

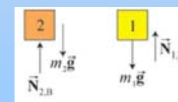


$$F_{\text{pivot}} - m_B g - N_{B,1} - N_{B,2} = 0$$

$$N_{1,B} - m_1 g = 0 \qquad N_{2,B} - m_2 g = 0$$

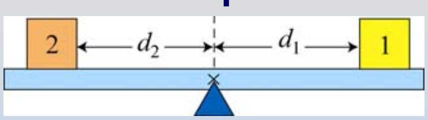
Third Law:

$$N_{1,B} = N_{B,1} \qquad N_{2,B} = N_{B,2}$$



Pivot force: $F_{\text{pivot}} = (m_B + m_1 + m_2)g$

Worked Example: Solution



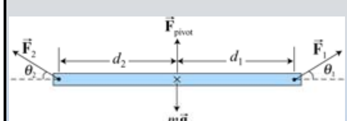
- Pivot force:

$$F_{\text{pivot}} = (m_B + m_1 + m_2)g = (2.0\text{ kg} + 0.3\text{ kg} + 0.6\text{ kg})(9.8\text{ m}\cdot\text{s}^{-2}) = 28.4\text{ N}$$

- Lever Law: $d_1 m_1 = d_2 m_2$

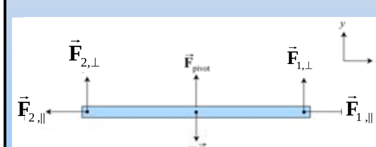
$$d_2 = d_1 m_1 / m_2 = (0.4\text{ m})(0.3\text{ kg} / 0.6\text{ kg}) = 0.2\text{ m}$$

Generalized Lever Law



$$\vec{\mathbf{F}}_1 = \vec{\mathbf{F}}_{1,\parallel} + \vec{\mathbf{F}}_{1,\perp}$$

$$\vec{\mathbf{F}}_2 = \vec{\mathbf{F}}_{2,\parallel} + \vec{\mathbf{F}}_{2,\perp}$$



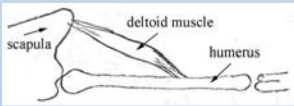
$$F_{1,\perp} = F_1 \sin\theta_1$$

$$F_{2,\perp} = F_2 \sin\theta_2$$

$$d_1 F_{1,\perp} = d_2 F_{2,\perp}$$

Table Problem : Forces and Torques on the Humerus

You are able to hold out your arm in an outstretched horizontal position thanks to the action of the deltoid muscle. Assume the humerus bone has a mass of *m*, the center of mass of the humerus is a distance *d* from the scapula, the deltoid muscle attaches to the humerus a distance *s* from the scapula and the angle the deltoid muscle makes with the horizontal is α. The scapula (shoulder blade) exerts an force on the humerus. The direction and magnitude of this force depends on the other parameters (that are fixed) *m, d,* α.



a) Draw a free body diagram for all the forces that are acting on the humerus. Indicate on your free body diagram your choice of unit vectors.

b) Choose a point about which to calculate the torques acting on the humerus. Explain why you decided on that point.

Optional: What is the tension T in the deltoid muscle? What are the vertical and horizontal components of the force exerted by the scapula (shoulder blade) on the humerus?