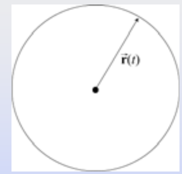
Circular Motion

8.01
W05D1

Position and Displacement

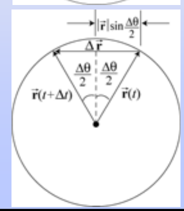
$\vec{\mathbf{r}}(t)$: position vector of an object moving in a circular orbit of radius R



$\Delta\vec{\mathbf{r}}(t)$: change in position between time t and time t+Δt

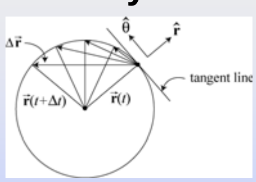
Position vector is changing in direction not in magnitude.

The magnitude of the displacement is the length of the chord of the circle:



$$\left|\Delta\vec{\mathbf{r}}\right| = 2R\sin(\Delta\theta/2)$$

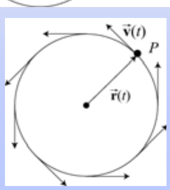
Direction of Velocity



Sequence of chord $\Delta\vec{r}$ directions approach direction of velocity as Δt approaches zero.

The direction of velocity is perpendicular to the direction of the position and tangent to the circular orbit.

Direction of velocity is constantly changing.



Small Angle Approximation

When the angle is small:

$$\sin\phi \approx \phi, \quad \cos\phi \approx 1$$

Power series expansion

$$\sin\phi = \phi - \frac{\phi^3}{3!} + \frac{\phi^5}{5!} - \ldots$$

$$\cos\phi = 1 - \frac{\phi^2}{2!} + \frac{\phi^4}{4!} - \ldots$$

Using the small angle approximation with $\phi = \Delta\theta/2$, the magnitude of the displacement is

$$\left|\Delta\vec{\mathbf{r}}\right| = 2R\sin(\Delta\theta/2) \approx R\Delta\theta$$

Speed and Angular Speed

The speed of the object undergoing circular motion is proportional to the rate of change of the angle with time:

$$v \equiv |\vec{\mathbf{v}}| = \lim_{\Delta t \to 0} \frac{|\Delta \vec{\mathbf{r}}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{R\,\Delta\theta}{\Delta t} = R \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = R\frac{d\theta}{dt} = R\omega$$

Angular speed: $\omega = \dfrac{d\theta}{dt}$ (units: $\mathrm{rad} \cdot \mathrm{s}^{-1}$)

Circular motion: Constant Speed, Period, and Frequency

In one period the object travels a distance equal to the circumference:

$$s = 2\pi R = vT$$

Period: the amount of time to complete one circular orbit of radius R

$$T = \frac{2\pi R}{v} = \frac{2\pi R}{R\omega} = \frac{2\pi}{\omega}$$

Frequency is the inverse of the period:

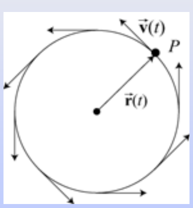
$$f = \frac{1}{T} = \frac{\omega}{2\pi} \quad (\text{units: } \mathrm{s}^{-1} \text{ or Hz})$$

Acceleration and Circular Motion

When an object moves in a circular orbit, the direction of the velocity changes and the speed may change as well.

For circular motion, the acceleration will always have a radial component (a_r) due to the change in direction of velocity

The acceleration may have a tangential component if the speed changes (a_t). When a_t =0, the speed of the object remains constant

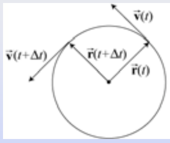


Concept Question: Coastal Highway

A sports car drives along the coastal highway at a constant speed. The acceleration of the car is

1. zero
2. sometimes zero
3. never zero
4. constant

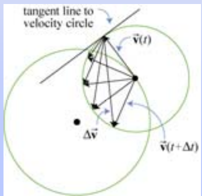
Direction of Radial Acceleration: Uniform Circular Motion



Sequence of chord directions $\Delta\vec{\mathbf{v}}$ approaches direction of radial acceleration as Δt approaches zero

Perpendicular to the velocity vector

Points radially inward



Change in Magnitude of Velocity:Uniform Circular Motion

Change in velocity:

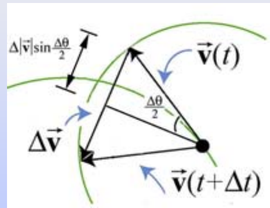
$$\Delta\vec{\mathbf{v}} = \vec{\mathbf{v}}(t+\Delta t) - \vec{\mathbf{v}}(t)$$

Magnitude of change in velocity:

$$\left|\Delta\vec{\mathbf{v}}\right| = 2v\sin(\Delta\theta/2)$$

Using small angle approximation

$$\left|\Delta\vec{\mathbf{v}}\right| \cong v\Delta\theta$$



Radial Acceleration: Constant Speed Circular Motion

Any object traveling in a circular orbit with a constant speed is always accelerating towards the center.

Direction of velocity is constantly changing.

Radial component of a_r *(minus sign indicates direction of acceleration points towards center)*

$$a_r = -\lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = -\lim_{\Delta t\to 0}\frac{v\Delta\theta}{\Delta t} = -v\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t} = -v\frac{d\theta}{dt} = -v\omega = -\frac{v^2}{R}$$

$$a_r = -\frac{v^2}{R} = -\omega^2 R$$

Tangential Acceleration

The *tangential acceleration* is the rate of change of the magnitude of the velocity

$$a_t = \lim_{\Delta t\to 0}\frac{\Delta v}{\Delta t} = R\lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t} = R\frac{d\omega}{dt} = R\frac{d^2\theta}{dt^2} = R\alpha$$

Angular acceleration: rate of change of angular velocity with time

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} \qquad (\text{units: rad}\cdot\text{s}^{-2})$$

Alternative forms of Magnitude of Radial Acceleration

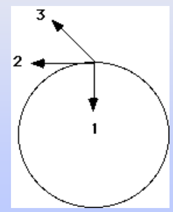
Parameters: speed v, angular speed ω, angular frequency f, period T

$$|a_r| = \frac{v^2}{R} = R\omega^2 = R(2\pi f)^2 = \frac{4\pi^2 R}{T^2}$$

Concept Question: Circular Motion

As the object speeds up along the circular path in a counterclockwise direction shown below, its acceleration points:

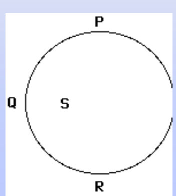
1. toward the center of the circular path.
2. in a direction tangential to the circular path.
3. outward.
4. none of the above.



Concept Question: Circular Motion

An object moves counter-clockwise along the circular path shown below. As it moves along the path its acceleration vector continuously points toward point *S*. The object

1. speeds up at *P*, *Q*, and *R*.
2. slows down at *P*, *Q*, and *R*.
3. speeds up at *P* and slows down at *R*.
4. slows down at *P* and speeds up at *R*.
5. speeds up at *Q*.
6. slows down at *Q*.
7. No object can execute such a motion.



Summary: Kinematics of Circular Motion

Arc length $s = R\theta$

Tangential velocity $v = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$

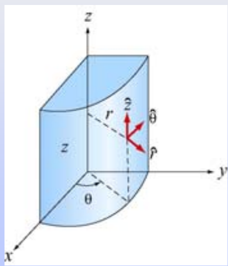
Tangential acceleration $a_t = \frac{dv}{dt} = R\frac{d^2\theta}{dt^2} = R\alpha$

Centripetal $a_r = v\omega = \frac{v^2}{R} = R\omega^2$

Cylindrical Coordinate System

Coordinates (r, θ, z)

Unit vectors $(\hat{\mathbf{r}}, \hat{\theta}, \hat{\mathbf{z}})$



Circular Motion: Vector Description

Use plane polar coordinates

Position $\vec{\mathbf{r}}(t) = R\,\hat{\mathbf{r}}(t)$

Velocity $\vec{\mathbf{v}}(t) = R\dfrac{d\theta}{dt}\,\hat{\theta}(t) = R\omega\,\hat{\theta}(t)$

Acceleration $\vec{\mathbf{a}} = a_r\,\hat{\mathbf{r}} + a_t\,\hat{\theta}$

$$a_t = r\alpha, \quad a_r = -r\omega^2 = -(v^2/r)$$

Modeling Problems: Circular Motion

Always has a component of acceleration pointing radially inward

May or may not have tangential component of acceleration

Draw Free Body Diagram for all forces

mv^2/r is not a force but mass times acceleration and does not appear on force diagram

Choose a sign convention for radial direction and check that signs for forces and acceleration are consistent

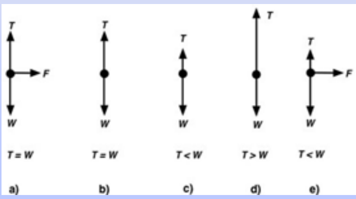
Newton's Second Law: Equations of Motion for Circular Motion

$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$

	physics	mathematics
$\hat{\mathbf{r}}$:	force decomposition	$= -mv^2/r$
$\hat{\boldsymbol{\theta}}$:	from	$= mrd^2\theta/dt^2$
$\hat{\mathbf{k}}$:	force diagram	$= 0$

Concept Question: Circular Motion and Force

A pendulum bob swings down and is moving fast at the lowest point in its swing. *T* is the tension in the string, *W* is the gravitational force exerted on the pendulum bob. Which free-body diagram below best represents the forces exerted on the pendulum bob at the lowest point? The lengths of the arrows represent the relative magnitude of the forces.



Concept Question: Car in a turn

You are a passenger in a racecar approaching a turn after a straight-away. As the car turns left on the circular arc at constant speed, you are pressed against the car door. Which of the following is true during the turn (assume the car doesn't slip on the roadway)?

1. A force pushes you away from the door.
2. A force pushes you against the door.
3. There is no force that pushes you against the door.
4. The frictional force between you and the seat pushes you against the door.
5. There is no force acting on you.
6. You cannot analyze this situation in terms of the forces on you since you are accelerating.
7. Two of the above.
8. None of the above.

Concept Question: Cart in a turn

A golf cart moves around a circular path on a level surface with decreasing speed. Which arrow is closest to the direction of the car's acceleration while passing the point P?

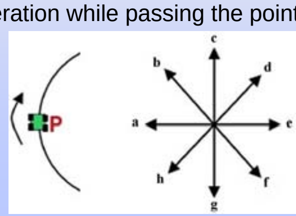
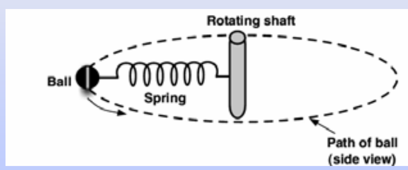


Table Problem: Experiment 2

One end of a spring is attached to the central axis of a motor. The axis of the motor is in the vertical direction. A small ball of mass m_2 is then attached to the other end of the spring. The motor rotates at a constant frequency *f*. Neglect the gravitational force exerted on the ball. Assume that the ball and spring rotate in a horizontal plane. The spring constant is *k*. Let r_0 denote the unstretched length of the spring.



(i) How long does it take the ball to complete one rotation?

(ii) What is the angular frequency of the ball in radians per sec?

(iii) What is the radius of the circular motion of the ball?