Two-Dimensional Rotational Kinematics

8.01
W10D1

Rigid Bodies

A **rigid body** is an extended object in which the distance between any two points in the object is constant in time.

Springs or human bodies are non-rigid bodies.

Rotation and Translation of Rigid Body

**Demonstration:** Motion of a thrown baton

Translational motion: external force of gravity acts on center of mass

Rotational Motion: object rotates about center of mass

**Recall: Translational Motion of the Center of Mass**

- Total momentum of system of particles
  \[ \mathbf{p}^{\text{total}} = m^{\text{total}} \mathbf{v}^{\text{cm}} \]
- External force and acceleration of center of mass
  \[ \mathbf{F}_{\text{ext}} = \frac{d\mathbf{p}^{\text{ext}}}{dt} = m^{\text{total}} \frac{d\mathbf{v}^{\text{cm}}}{dt} = m^{\text{total}} \mathbf{a}^{\text{cm}} \]
**Main Idea: Rotation of Rigid Body**

Torque produces angular acceleration about center of mass

\[ \tau_{\text{total}}^{\text{cm}} = I_{\text{cm}} \alpha_{\text{cm}} \]

\( I_{\text{cm}} \) is the moment of inertia about the center of mass

\( \alpha_{\text{cm}} \) is the angular acceleration about center of mass

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**Rotation of Rigid Body**


**Two-Dimensional Rotation**

- **Fixed axis rotation:**
  Disc is rotating about axis passing through the center of the disc and is perpendicular to the plane of the disc.

- **Plane of motion is fixed:**
  For straight line motion, bicycle wheel rotates about fixed direction and center of mass is translating

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**Rotational Kinematics for Fixed Axis Rotation**

A point like particle undergoing circular motion at a non-constant speed has

1. An angular velocity vector
2. An angular acceleration vector

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**Fixed Axis Rotation: Angular Velocity**

Angle variable

\[ \theta \] [rad]

Angular velocity

\[ \omega = \frac{d\theta}{dt} \] [rad/s]

Vector Component

<table>
<thead>
<tr>
<th>( \omega )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega &gt; 0 ), direction ( +\hat{k} )</td>
</tr>
<tr>
<td>( \omega &lt; 0 ), direction ( -\hat{k} )</td>
</tr>
</tbody>
</table>

Magnitude

\[ |\omega| = \left| \frac{d\theta}{dt} \right| \]
Fixed Axis Rotation: Angular Acceleration

Angular acceleration: \( \mathbf{\alpha} = \alpha \mathbf{k} = \frac{d^2 \theta}{dt^2} \mathbf{k} \)

SI unit \([\text{rad} \cdot \text{s}^{-2}]\)

Vector: \( \mathbf{\alpha} \)

Component:

- Magnitude: \( |\mathbf{\alpha}| = \left| \frac{d\omega}{dt} \right| \)
- Direction:
  - \( \frac{d\omega}{dt} > 0 \), direction \( +\mathbf{k} \)
  - \( \frac{d\omega}{dt} < 0 \), direction \( -\mathbf{k} \)

Concept Question: Angular Speed

Object A sits at the outer edge (rim) of a merry-go-round, and object B sits halfway between the rim and the axis of rotation. The merry-go-round makes a complete revolution once every thirty seconds. The magnitude of the angular velocity of Object B is

1. half the magnitude of the angular velocity of Object A.
2. the same as the magnitude of the angular velocity of Object A.
3. twice the the magnitude of the angular velocity of Object A.
4. impossible to determine.

Rotational Kinematics: Constant Angular Acceleration

The angular quantities \( \theta, \omega, \) and \( \alpha \)

are exactly analogous to the quantities \( x, v, \) and \( a \),

for one-dimensional motion, and obey the same type of integral relations

\( \omega(t) - \omega_0 = \int \alpha(t') dt' \), \( \theta(t) - \theta_0 = \int \omega(t') dt' \).

Constant angular acceleration:

\[
\begin{align*}
\omega(t) &= \omega_0 + \alpha t \\
\theta(t) &= \theta_0 + \omega_0 t + \frac{1}{2} \alpha t^2
\end{align*}
\]

\[
\Rightarrow (\omega(t))^2 = \omega_0^2 + 2\alpha (\theta(t) - \theta_0)
\]

Table Problem: Rotational Kinematics

A turntable is a uniform disc of mass \( m \) and a radius \( R \). The turntable is spinning initially at a constant frequency \( f \). The motor is turned off and the turntable slows to a stop in \( t \) seconds with constant angular acceleration.

a) What is the direction and magnitude of the initial angular velocity of the turntable?
b) What is the direction and magnitude of the angular acceleration of the turntable?
c) What is the total angle in radians that the turntable spins while slowing down?
Angular Momentum of a Point Particle

- Point particle of mass $m$ moving with a velocity $\vec{v}$
- Momentum $\vec{p} = m\vec{v}$
- Fix a point $S$
- Vector $\vec{r}_S$ from the point $S$ to the location of the object
- Angular momentum about the point $S$
- SI Unit $[\text{kg} \cdot \text{m}^2 \cdot \text{s}^{-1}]$

$$L_S = \vec{r}_S \times \vec{p}$$

Cross Product: Angular Momentum of a Point Particle

Direction

Right Hand Rule

Cross Product: Angular Momentum of a Point Particle

Magnitude:

- a) moment arm $r_{S,\perp} = |\vec{r}_S| \sin \theta$
- b) Perpendicular momentum $p_{S,\perp} = |\vec{p}| \sin \theta$

Example Problem: Angular Momentum and Cross Product

A particle of mass $m = 2$ kg moves with a uniform velocity $\vec{v} = 3.0 \text{ m/s} \hat{i} + 3.0 \text{ m/s} \hat{j}$

At time $t$, the position vector of the particle with respect of the point $S$ is $\vec{r}_S = 2.0 \text{ m} \hat{i} + 3.0 \text{ m} \hat{j}$

Find the direction and the magnitude of the angular momentum about the origin, (the point $S$) at time $t$. 
Solution: Angular Momentum and Cross Product

The angular momentum vector of the particle about the point \( S \) is given by:

\[
\mathbf{L}_p = \mathbf{r}_S \times \mathbf{p} - \mathbf{r}_S \times m \mathbf{v}
\]

\[
= \left( 2.0 \text{ m} \hat{i} + 3.0 \text{ m} \hat{j} \right) \times \left( 2 \text{ kg} \left( 3.0 \text{ m} \cdot \text{s}^{-1} \hat{i} + 3.0 \text{ m} \cdot \text{s}^{-1} \hat{j} \right) \right)
\]

\[
= \left( 0 + 2 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \hat{k} - 18 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \left( -\hat{k} \right) \right) \times \hat{\theta}
\]

\[
= -6.0 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1} \hat{\theta}.
\]

The direction is in the negative \( \hat{\theta} \) direction, and the magnitude is

\[
\left| \mathbf{L}_p \right| = 6.0 \text{ kg} \cdot \text{m}^2 \cdot \text{s}^{-1}.
\]

Table Problem: Angular Momentum and Circular Motion

Consider a point particle of mass \( m \) moving in a circle of radius \( R \) with velocity \( \mathbf{v} = R \omega \hat{\theta} \). Find the direction and magnitude of the angular momentum about the center of a circle in terms of the radius \( R \), mass \( m \), and angular speed \( \omega \).

Angular Momentum and Circular Motion of a Point Particle

Fixed axis of rotation: \( z \)-axis

Angular velocity \( \hat{\omega} = \omega \hat{k} \)

Velocity

\( \mathbf{v} = \hat{\omega} \times \mathbf{r} = \omega \hat{k} \times R \hat{r} = R \omega \hat{\theta} \)

Angular momentum about the point \( S \)

\[
\mathbf{L}_S = \mathbf{r}_S \times \mathbf{p} = \mathbf{r}_S \times m \mathbf{v} = R m \omega \hat{k} = R m \omega \hat{k} = m R^2 \omega \hat{k}.
\]

Rigid Body Kinematics for Fixed Axis Rotation

Body rotates with angular velocity \( \omega \) and angular acceleration \( \alpha \).
Divide Body into Small Elements

Body rotates with angular velocity, $\omega$

Angular acceleration, $\alpha$

Individual elements of mass, $\Delta m_i$

Radius of orbit, $r_{\perp, i}$

Tangential velocity, $v_{\text{tan},i} = r_{\perp, i} \omega$

Tangential acceleration, $a_{\text{tan},i} = r_{\perp, i} \alpha$

Radial Acceleration, $a_{\text{rad},i} = \frac{v_{\text{tan},i}^2}{r_{\perp, i}^2}$

Rotational Kinetic Energy and Moment of Inertia

Rotational kinetic energy about axis passing through $S$

$$K_{\text{rot}} = \frac{1}{2} \sum_{i=1}^{N} \Delta m_i \left( r_{\perp, i} \right)^2 \omega^2$$

Moment of Inertia about $S$: $I_s = \sum \Delta m_i \left( r_{\perp, i} \right)^2$

SI Unit: [kg \cdot m^2]

Continuous body:

$$I_s = \int_{\text{body}} dm \left( r_{\perp, i} \right)^2$$

Rotational Kinetic Energy:

$$K_{\text{rot}} = \frac{1}{2} \int_{\text{body}} dm \left( r_{\perp, i} \right)^2 \omega^2 = \frac{1}{2} I_s \omega^2$$

Discussion: Moment of Inertia

How does moment of inertia compare to the total mass and the center of mass?

Different measures of the distribution of the mass.

Total mass: scalar

$$m_{\text{total}} = \int_{\text{body}} dm$$

Center of Mass: vector (three components)

$$\mathbf{R}_{\text{cm}} = \frac{1}{m_{\text{total}}} \int_{\text{body}} \mathbf{r} \, dm$$

Moment of Inertia about axis passing through $S$: (nine possible moments)

$$I_s = \int_{\text{body}} dm \left( r_{\perp, i} \right)^2$$

Concept Question: Collision

An object of mass $m$ with velocity $\vec{v}$ moving in a straight line collides elastically with an identical object. The second object is initially at rest and is attached at one end to a string of length $l$ and negligible mass. The other end of the string is fixed (at the point $S$). After the collision the second object undergoes circular motion with angular speed $\omega$. The kinetic energy of the second object after the collision is

1. $\frac{1}{4} m l^2 \omega^2$
2. $\frac{1}{2} m l^2 \omega^2$
3. $\frac{3}{4} m l^2 \omega^2$
4. $\frac{1}{4} m l^2 \omega^2$
5. $\frac{1}{2} m l^2 \omega^2$
6. $\frac{1}{4} m l^2 \omega$
Strategy: Calculating Moment of Inertia

Step 1: Identify the axis of rotation
Step 2: Choose a coordinate system
Step 3: Identify the infinitesimal mass element dm.
Step 4: Identify the radius, \( r \), of the circular orbit of the infinitesimal mass element dm.
Step 5: Set up the limits for the integral over the body in terms of the physical dimensions of the rigid body.
Step 6: Explicitly calculate the integrals.

Example: Moment of Inertia of a Disc

Consider a thin uniform disc of radius \( R \) and mass \( m \). What is the moment of inertia about an axis that pass perpendicular through the center of the disc?

\[
I_{cm} = \int (r^2) dm = \frac{M}{\pi R^2} \int_0^{2\pi} \int_0^R r^2 r \, dr \, d\theta
\]

\[
I_{cm} = \frac{M}{\pi R} \left( \frac{4}{3} R^3 \right) = \frac{2}{3} M R^2
\]

Table Problem: Moment of Inertia of a Rod

Consider a thin uniform rod of length \( L \) and mass \( m \). Calculate the moment of inertia about an axis that passes perpendicular through the center of mass of the rod.

Parallel Axis Theorem

- Rigid body of mass \( m \).
- Moment of inertia \( I_{cm} \) about axis through center of mass of the body.
- Moment of inertia \( I_p \) about parallel axis through point \( S \) in body.
- \( d_{csm} \) perpendicular distance between two parallel axes.

\[
I_p = I_{cm} + m d_{csm}^2
\]
Summary: Moment of Inertia

Moment of Inertia about S:

\[ I_s = \sum_{i=1}^{N} \Delta m_i (r_{i,S})^2 \]

Examples: Let S be the center of mass
- rod of length l and mass m: \[ I_{cm} = \frac{1}{12} ml^2 \]
- disc of radius R and mass m: \[ I_{cm} = \frac{1}{2} mR^2 \]

Parallel Axis theorem:
\[ I_s = I_{cm} + md_s^2 \]

Concept Question: Kinetic Energy

A disk with mass M and radius R is spinning with angular speed \( \omega \) about an axis that passes through the rim of the disk perpendicular to its plane. Moment of inertia about cm is \( (1/2)MR^2 \). Its total kinetic energy is:

1. \( (1/4)MR^2 \omega^2 \)
2. \( (1/2)MR^2 \omega^2 \)
3. \( (3/4)MR^2 \omega^2 \)
4. \( (1/4)MR^2 \omega^2 \)
5. \( (1/2)MR^2 \omega^2 \)
6. \( (1/4)MR^2 \omega^2 \)

Summary: Fixed Axis Rotation

Kinematics

<table>
<thead>
<tr>
<th>Angle variable</th>
<th>( \theta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular velocity</td>
<td>( \omega = d\theta / dt )</td>
</tr>
<tr>
<td>Angular acceleration</td>
<td>( \alpha = d^2\theta / dt^2 )</td>
</tr>
<tr>
<td>Mass element</td>
<td>( \Delta m_i )</td>
</tr>
<tr>
<td>Radius of orbit</td>
<td>( r_{i,S} )</td>
</tr>
<tr>
<td>Moment of inertia</td>
<td>( I_s = \sum_{i=1}^{N} \Delta m_i (r_{i,S})^2 )</td>
</tr>
<tr>
<td>Parallel Axis Theorem</td>
<td>( I_s = Md_s^2 + I_{cm} )</td>
</tr>
</tbody>
</table>

Angular Momentum for Fixed Axis Rotation

Angular Momentum about the point S
\[ \vec{L}_{x,y,z} = \vec{r}_{x,y,z} \times \vec{p}_{x,y,z} = (r_x \hat{i} + r_y \hat{j} + r_z \hat{k}) \times (p_{x,y,z} \hat{\theta}) \]

Tangential component of momentum

\[ p_{x,y,z} = \Delta m \hat{r}_{x,y,z} \times \vec{r}_{x,y,z} \times \hat{\theta} \]

z-component of angular momentum about S:
\[ \vec{L}_{z} = \vec{r}_{x,y,z} \times \vec{p}_{x,y,z} = \sum_{i=1}^{N} \Delta m_i r_{i,S} \hat{j} \times \vec{r}_{i,S} \times \hat{k} \]

\[ L_z = \sum_{i=1}^{N} \Delta m_i r_{i,S} \hat{j} \times \vec{r}_{i,S} \times \hat{k} = I_z \hat{\theta} \]
Concept Question: Angular Momentum

A dumbbell is rotating at a constant angular speed about its center. Compared to the dumbbell’s angular momentum about its center A, its angular momentum about point B (as shown in the figure) is

1. bigger.
2. the same.
3. smaller.

Concept Question: Angular momentum

A disk with mass $M$ and radius $R$ is spinning with angular velocity $\omega$ about an axis that passes through the rim of the disk perpendicular to its plane. The magnitude of its angular momentum is:

1. $\frac{1}{4} MR^2 \omega^2$
2. $\frac{1}{2} MR^2 \omega^2$
3. $\frac{3}{2} MR^2 \omega^2$
4. $\frac{1}{4} MR^2 \omega$
5. $\frac{1}{2} MR^2 \omega$
6. $\frac{3}{2} MR^2 \omega$