2-Dimensional Motion: Rotation and Translation

8.01
W13D1 Fall 2006

Introduction: Rotation and Translation of Rigid Body

Translational motion: the total external force of gravity acts on center-of-mass

\[ F_{ext} = m \ddot{v}_G = m \ddot{v}_m - m \ddot{a}_m \]

Rotational Motion: object rotates about center-of-mass. Note that the center-of-mass may be accelerating

Introduction: Rotation about the Center-of-Mass of a Rigid Body

The total external torque produces an angular acceleration about the center-of-mass

\[ \alpha_{cm} = \frac{d\omega_{cm}}{dt} = \frac{dL_{cm}}{dt} \]

\( I_{cm} \) is the moment of inertia about the center-of-mass

\( \alpha_{cm} \) is the angular acceleration about the center-of-mass

\( \bar{L}_{cm} \) is the angular momentum about the center-of-mass

Two Reference Systems

'Laboratory reference frame' is a coordinate system fixed to the lab (the center-of-mass of the rigid body is translating in this frame)

'Center-of-mass reference frame' is a coordinate system that moves with the center-of-mass of the body which may or may not be accelerating with respect to the laboratory frame

\[ \bar{R} = \bar{R}_{cm} + \bar{r}_{cm} \]

\[ \bar{v} = \bar{v}_{cm} + \dot{\bar{r}}_{cm} \]

\[ \ddot{\bar{a}} = \ddot{\bar{a}}_{cm} + \dot{\bar{r}}_{cm} \]
Questions: Torque about the Center of Mass

What forces do we use to calculate the torque about the center of mass?

$$\tau_{\text{cm}} = \sum_{i=1}^{N} r_{i} \times F_{i}$$

Do we use the same forces in the laboratory and center-of-mass reference frames?

Where does the gravitational force act?

Answer: The gravitational force acts at the center of mass as does the 'fictitious force' that is actually the result of the effect of the acceleration of the center of mass. Since they act at the center of mass, neither contributes to the torque. So we use the same forces that are observed in the laboratory frame to calculate the torque about the center-of-mass.

Torque due to Uniform Gravitational Force

The total torque on a rigid body due to the gravitational force can be determined by placing all the gravitational force at the center-of-mass of the object.

$$\tau_{\text{grav}} = \sum_{i=1}^{N} r_{i} \times F_{i} = \sum_{i=1}^{N} r_{i} \times m \vec{g} = \sum_{i=1}^{N} m_{i} \vec{r}_{i} \times \vec{g} = \left( \frac{1}{m} \sum_{i=1}^{N} m_{i} \vec{r}_{i} \right) \times m \vec{g}$$

Torque for Rotation and Translation

The total torque about S is given by

$$\tau = \tau_{\text{ext}} + \tau_{\text{cm}}$$

where the first term torque about S due to the total external force acting at the center-of-mass and the second term is torque about the center-of-mass is only due to the forces as seen in the laboratory frame.

$$\tau_{\text{ext}} = R_{\text{ext}} \times F_{\text{ext}}$$

$$\tau_{\text{cm}} = \sum_{i=1}^{N} r_{i} \times F_{i}$$

The total external torque produces an angular acceleration about the center-of-mass

$$\tau_{\text{ext}} = I_{\text{cm}} \alpha_{\text{cm}} = \frac{d^2 \theta}{dt^2}$$

Concept Question: Rotation and Translation

Two disks are separated by a spindle of smaller diameter. A string is wound around the spindle and pulled gently. Which positions of the string cause the assembly to roll to the right?

1) Only A
2) Only B
3) Only C
4) A and B
5) B and C
6) A and B and C
7) None of the configurations shown
Demo: Giant Yo-Yo

Demo: Descending and Ascending Yo-Yo

$$M_{\text{total yo-jo}} = 435 \text{ g}$$
$$R_{\text{outer}} \approx 6.3 \text{ cm}$$
$$R_{\text{inner}} \approx 4.9 \text{ cm}$$

$$I_{cm} \approx \frac{1}{2} MR_{outer}^2 + \frac{1}{2} R_{inner}^2 = 1.385 \times 10^4 \text{ g} \cdot \text{cm}^2$$

Rotational Work-Kinetic Energy Theorem

Kinetic energy of rotation about center-of-mass

$$W_{\text{rot}} = \frac{1}{2} I_{cm} \omega_{cm}^2 - \frac{1}{2} I_{cm} \omega_{cm}^2 = K_{\text{rot}, f} - K_{\text{rot}, i} = \Delta K_{\text{rot}}$$

Rotation and translation

$$W_{\text{total}} = \Delta K_{\text{trans}} + \Delta K_{\text{rot}}$$

$$W_{\text{total}} = \Delta K_{\text{trans}} + \Delta K_{\text{rot}} = \left( \frac{1}{2} mv_{cm,f}^2 - \frac{1}{2} mv_{cm,i}^2 \right) + \left( \frac{1}{2} I_{cm} \omega_{cm,f}^2 - \frac{1}{2} I_{cm} \omega_{cm,i}^2 \right)$$

Concept Question: Rotation and Translation

Two cylinders of the same size and mass roll down an incline, starting from rest. Cylinder A has most of its mass concentrated at the rim, while cylinder B has most of its mass concentrated at the center.

Which has more total kinetic energy at the bottom?

1. A
2. B
3. Both have the same
Concept Question: Rotation and Translation

Two cylinders of the same size and mass roll down an incline, starting from rest. Cylinder A has most of its mass concentrated at the rim, while cylinder B has most of its mass concentrated at the center. Which reaches the bottom first?

1. A
2. B
3. Both have the same

Demo: Rolling Cylinders

Concept Question Problem: Cylinder Rolling Down an Inclined Plane

Simple Pendulum

Simple Pendulum: bob of mass $m$ hanging from end of massless string string pivoted at $S$.

- Torque about $S$: $\tau_l = \ell \times mg \times (-\sin \theta \hat{\theta} + \cos \theta) = -mg \sin \theta \hat{k}$
- Angular acceleration: $\ddot{\theta} = \frac{d^2 \theta}{dt^2}$
- Moment of inertia of a point mass about $S$: $I_\ell = ml^2$
- Rotational Law of Motion: $\tau_l = I_\ell \ddot{\theta}$
- Simple harmonic oscillator equation: $-mg \sin \theta = ml^2 \frac{d^2 \theta}{dt^2}$
Simple Pendulum: Small Angle Approximation

- Angle of oscillation is small: \( \sin \theta \approx \theta \)
- Simple harmonic oscillator: \( \frac{d^2\theta}{dt^2} = \frac{g}{l} \theta \)
- Analog to spring equation: \( \frac{d^2x}{dt^2} = \frac{k}{m} x \)
- Angular frequency of oscillation: \( \omega \approx \sqrt{\frac{g}{l}} \)
- Period: \( T \approx \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l}{g}} \)

Physical Pendulum

- Pendulum pivoted about point S
- Gravitational force acts on center of mass
- Center of mass distance from the pivot point: \( l_{cm} \)

Physical Pendulum

- Rotational dynamical equation: \( \vec{\tau} = I \vec{\alpha} \)
- Small angle approximation: \( \sin \theta \approx \theta \)
- Equation of motion: \( \frac{d^2\theta}{dt^2} = \frac{l_{cm}g}{l_i} \theta \)
- Angular frequency: \( \omega = \sqrt{\frac{l_{cm}g}{l_i}} \)
- Period: \( T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{l_i}{l_{cm}mg}} \)

Rotation and Translation

- The center-of-mass of the physical pendulum is undergoing non-uniform circular motion.
- The body is also rotating about the center-of-mass.
- The physical pendulum is a special case (pivoted about a point) of a more general motion of a rigid body.
Concept Question: Physical Pendulum
A physical pendulum consists of a uniform rod of length \( l \) and mass \( m \) pivoted at one end. A disk of mass \( m_1 \) and radius \( a \) is fixed to the other end. Suppose the disk is now mounted to the rod by a frictionless bearing so that it is perfectly free to spin. Does the period of the pendulum

1. increase?
2. stay the same?
3. decrease?

Demo: Identical Pendulums, Different Periods

Single pivot: body rotates about center of mass. Double pivot: no rotation about center of mass.

Appendices
1. Torque About the Center-of-Mass
2. Angular Momentum for Rotation and Translation
3. Torque for Rotation and Translation

Questions: Torque about the Center of Mass
What forces do we use to calculate the torque about the center of mass?

\[
\tau_{cm} = \sum \vec{r}_m \times \vec{F}_j
\]

Do we use the same forces in the laboratory and center-of-mass reference frames?

Where does the gravitational force act?

Answer: The gravitational force acts at the center of mass as does the 'fictitious force' that is actually the result of the effect of the acceleration of the center of mass. Since they act at the center of mass, neither contributes to the torque. So we use the same forces that are observed in the laboratory frame to calculate the torque about the center-of-mass.
Two Reference Systems

"Laboratory reference frame" is a coordinate system fixed to the lab (the center-of-mass of the rigid body is translating in this frame).

"Center-of-mass reference frame" is a coordinate system that moves with the center-of-mass of the body which may or may not be accelerating with respect to the laboratory frame.

$\mathbf{F} = \mathbf{R} + \mathbf{F}_{\text{cm}}$
$\mathbf{V} = \mathbf{V}_{\text{cm}}$
$\mathbf{a} = \mathbf{a}_{\text{cm}}$

Torque due to Uniform Gravitational Force

The total torque on a rigid body due to the gravitational force can be determined by placing all the gravitational force at the center-of-mass of the object.

$\mathbf{r}_{\text{cm}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{g} = \sum_{i=1}^{n} m_i \mathbf{r}_i \times \mathbf{g}$
$= \left( \frac{1}{m_{\text{cm}}} \sum_{i=1}^{n} m_i \mathbf{r}_i \right) \times m_{\text{cm}} \mathbf{g}$
$= \mathbf{R}_{\text{cm}} \times m_{\text{cm}} \mathbf{g}$

Fictitious Forces in Non-Inertial Reference Frames

Newton's Second Law for a mass element $m$ in the body in the laboratory frame is

$\mathbf{F} = m\mathbf{a}$
$\mathbf{F} = m(\dot{\mathbf{A}} + \ddot{\mathbf{A}})$

In the center-of-mass frame that is accelerating with respect to the laboratory frame, define the total force acting on the mass element $m$ by

$\mathbf{F}' = m\ddot{\mathbf{A}}_{\text{cm}}$

Then the total force in the center-of-mass frame is related to the total force in the laboratory frame by

$\mathbf{F}' = \mathbf{F} - m\ddot{\mathbf{A}} = m\ddot{\mathbf{A}}_{\text{cm}}$

The quantity $F_{\text{cm}} = -m\ddot{\mathbf{A}}_{\text{cm}}$ is called the fictitious force and is just the result of the acceleration of the center-of-mass frame.

Fictitious Forces, Equivalent Gravitational Fields, and Torque

When the center-of-mass is moving with constant acceleration

$\frac{d\mathbf{A}}{dt} = \mathbf{0}$

Then the fictitious force acts like a uniform gravitational field whose acceleration vector is given by

$\mathbf{g} = -\mathbf{A}$

The total torque on a rigid body about a point $S$ due to the fictitious force arising from a uniform acceleration can be determined by placing the fictitious force at the center-of-mass of the object.

$\mathbf{r}_{\text{fict}} = \sum_{i=1}^{n} \mathbf{r}_i \times \mathbf{F}_{\text{fict},i} = \sum_{i=1}^{n} m_i \mathbf{r}_i \times -m\ddot{\mathbf{A}}_{\text{cm}} = -\mathbf{R}_{\text{cm}} \times m_{\text{cm}} \mathbf{A}$
Torque due to Gravitational Force and Fictitious Force about Center-of-mass

Therefore if we calculate the torque about the center-of-mass due to the forces acting in the center-of-mass reference frame, both the gravitational force and the fictitious forces do not contribute to the torque since they both act at the center-of-mass.

\[ \tau_{cm} = 0 \]

\[ \tau_{G} = 0 \]

\[ \tau_{S,grav} = 0 \]

Angular Momentum for Rotation and Translation

The angular momentum for a rotating and translating object is given by (see next two slides for details of derivation)

\[ \mathbf{L} = \mathbf{R} \times \mathbf{p} - \sum_{i=1}^{N} m_{i} \mathbf{r}_{i} \times \mathbf{v}_{i} \]

The first term in the expression for angular momentum about \( S \) arises from treating the body as a point mass located at the center-of-mass, moving with a velocity equal to the center-of-mass velocity.

\[ \mathbf{L}_{S} = \sum_{i=1}^{N} m_{i} \mathbf{r}_{i} \times \mathbf{v}_{i} \]

The second term is the angular momentum about the center-of-mass,

\[ \mathbf{L}_{cm} = \sum_{i=1}^{N} m_{i} \mathbf{r}_{i} \times \mathbf{v}_{i} \]

Derivation: Angular Momentum for Rotation and Translation

The angular momentum for a rotating and translating object is given by

\[ \mathbf{L} = \mathbf{R} \times \mathbf{p} + \sum_{i=1}^{N} m_{i} \mathbf{r}_{i} \times \mathbf{v}_{i} \]

The position and velocity with respect to the center-of-mass reference frame of each mass element is given by

\[ \mathbf{r}_{i} = \mathbf{R}_{cm} \mathbf{r}_{i} + \mathbf{r}_{i} \]

\[ \mathbf{v}_{i} = \mathbf{v}_{cm} + \mathbf{v}_{i} \]

So the angular momentum can be expressed as

\[ \mathbf{L} = \mathbf{R} \times \mathbf{p} + \sum_{i=1}^{N} m_{i} \mathbf{r}_{i} \times \mathbf{v}_{i} \]

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The two middle terms in the above expression vanish because in the center-of-mass frame, the position of the center-of-mass is at the origin, and the total momentum in the center-of-mass frame is zero.

\[ \frac{1}{m} \sum_{i=1}^{N} m_{i} \mathbf{v}_{cm} = \mathbf{0} \]

Then then angular momentum about \( S \) becomes

\[ \mathbf{L}_{S} = \mathbf{R} \times \sum_{i=1}^{N} m_{i} \mathbf{v}_{i} + \sum_{i=1}^{N} m_{i} \mathbf{r}_{i} \times \mathbf{v}_{i} \]

The momentum of system is

\[ \mathbf{p} = \sum_{i=1}^{N} m_{i} \mathbf{v}_{i} \]

So the angular momentum about \( S \) is

\[ \mathbf{L}_{S} = \mathbf{R} \times \mathbf{p} + \sum_{i=1}^{N} m_{i} \mathbf{r}_{i} \times \mathbf{v}_{i} \]
Earth’s Motion about Sun: Orbital Angular Momentum

For a body undergoing orbital motion like the earth orbiting the sun, the two terms can be thought of as an orbital angular momentum about the center-of-mass of the earth-sun system, denoted by $S$

$$L_{orb} = R_{orbi} \times \mathbf{p}_{orbi} = r_e m_v \mathbf{k}$$

Spin angular momentum about center-of-mass of earth

$$L_{spin} = \int_m \mathbf{r}_e \times m \mathbf{v}_e \, dm$$

Total angular momentum about $S$

$$L_{total} = L_{orb} + L_{spin}$$

Conservation of Angular Momentum

No external torques about point $S$: angular momentum about $S$ is constant

$$\theta = \int \frac{dL_{total}}{dt}$$

Table Problem: Bicycle Wheel

Consider a bicycle wheel of radius $R$ and mass $m$ with moment of inertia $I_{cm}$ about an axis passing perpendicular to the plane of the wheel and through the center-of-mass. The bicycle wheel is initially spinning with angular velocity $\omega_0$ about the center-of-mass. The wheel is lowered to the ground without bouncing. As soon as the wheel touches the level ground, the wheel starts to accelerate forward until it begins to roll without slipping with a final angular velocity $\omega_f$ and center-of-mass velocity $v_{cm}$. What is the velocity of the center-of-mass when the wheel rolls without slipping?

Torque and Angular Momentum for Rotation and Translation

The torque about a point $S$ is the time derivative of the angular momentum about $S$

$$\tau_s = \frac{dL_s}{dt}$$

$$\tau_s = \frac{d}{dt}(R_{orbi} \times \mathbf{p}_{orbi}) + \frac{d}{dt}(\int m \mathbf{r} \times \mathbf{v} \, dm) + \frac{d}{dt}(\sum m \mathbf{v} \times (\frac{d}{dt}m \mathbf{v} \, dm))$$

Once again the first and third terms vanish because

$$\frac{d}{dt}(R_{orbi} \times \mathbf{p}_{orbi}) = \mathbf{0} \quad \frac{d}{dt}(\sum m \mathbf{v} \times (\frac{d}{dt}m \mathbf{v} \, dm)) = \mathbf{0}$$

So the torque about $S$ becomes,

$$\tau_s = \frac{d}{dt}(\int m \mathbf{r} \times \mathbf{v} \, dm)$$
The torque about a point $S$ is
\[ \vec{\tau}_S = \vec{R}_S \times \sum \vec{F}_{ext,i} \]
Recall that the external force is the time change of the momentum of the center-of-mass,
\[ \vec{p}_{cm} = \sum \vec{m}_i \vec{v}_i \]
So the first term is the torque about $S$ due to the total external force acting at the center-of-mass
\[ \vec{\tau}_{ext} = \vec{R}_S \times \vec{p}_{cm} \]

When we sum the torques over all the elements in the body, the fictitious forces act at the center-of-mass, so the torque from these fictitious forces is zero, so, the torque about the center-of-mass is only due to the forces as seen in the laboratory frame.
\[ \vec{\tau}_{cm} = \sum \vec{R}_{cm,i} \times \vec{F}_{cm,i} = \sum \vec{R}_{cm,i} \times (\vec{F}_{cm,i} + \vec{m}_i \vec{a}_i) \]
\[ = \sum \vec{R}_{cm,i} \times \vec{F}_{cm,i} + \sum \vec{R}_{cm,i} \times \vec{m}_i \vec{a}_i \]
\[ = \sum \vec{R}_{cm,i} \times \vec{F}_{cm,i} - \sum \vec{R}_{cm,i} \times \vec{m}_i \vec{a}_i \]
\[ = \sum \vec{R}_{cm,i} \times \vec{F}_{cm,i} - \sum \vec{F}_{cm,i} \times \vec{R}_{cm,i} \]

The time derivative that appears in the second term in the above expression is the time derivative of the momentum of a mass element in the center-of-mass-frame is the force acting on that element which include both inertial and fictitious force,
\[ \frac{d}{dt} \vec{m}_i \vec{v}_{cm,i} = \vec{F}_{cm,i} \]

The torque about the center-of-mass is then
\[ \vec{\tau}_{cm} = \vec{R}_{cm} \times \sum \vec{m}_i \vec{v}_{cm,i} = \vec{R}_{cm} \times \sum \vec{F}_{cm,i} \]