Problem 1: Estimation (10 points)

Your car has a flat and you try to loosen the lugs on the wheel with a tire iron shown in the figure on the left below. You can’t budge the lugs but then you try the 4-way wrench shown in the figure on the right and the lugs loosen. Based on the figures below, estimate how much torque you needed to apply to loosen the lugs.

Possible Answer: There are many considerations in this problem, and if you’ve had some experience with changing flat tires, you’re aware of what they are. Your experience may suggest that the 4-way is overall easier to use.

Start by keeping in mind that the car should be up on a jack, and many jacks don’t supply good overall stability. Applying too much net force, up or down, on the wheel could cause the car to fall off the jack. As an example, for a 1983 VW Rabbit (since departed), the car could be lifted off the jack. Typically, a person can exert an upward force of 2-3 times the person’s weight. So, using the simple wrench on the left and pulling up with both hands warrants caution. Pushing down, the maximum downward force exerted can’t exceed the person’s weight. As an estimate of the net torque, use a force with magnitude equal to your weight and a moment arm of about half a meter. For me, that’s about 500 N-m. Of course, the angle between the applied force and the wrench should be 90°; the wrench arm should be horizontal.

For the 4-way wrench, assume the same magnitude of force applied on each of the two arms used to turn the wrench, one force directed up and the other down. Ideally, this would mean that you apply no net force the car, and falling off the jack is less likely. Also, you would be able to position yourself more or less symmetric about the wheel, making the process a bit more comfortable. (But the forces on your feet wouldn’t be the same – see Problem 3.) The forces you exert then form a “couple,” and the torque would be the product of the applied force and the moment arm, times 2 for the two arms of the wrench. From the figure, it looks like the moment arms are about the same as for the
simple wrench, so the net torque is roughly your weight times the moment arm of about half a meter times 2, or 1000 N⋅m; much easier to undo the lug nuts.

**Problem 2: Torque on the right leg (10 points)** We want to find the magnitude of the tension in the hip abductor muscles \( \mathbf{T} \) shown in the figure below. The direction \( \alpha \) is given. After drawing your free body diagram for the right leg, you are trying to decide what point to compute the torques about. Explain the disadvantages /advantages if you chose each of the points listed below.

![Diagram of the right leg with labeled forces and moments](image)

a) The center of mass of the leg where the gravitational force \( m \mathbf{g} \) acts
b) The contact point with the ground where the normal force \( \mathbf{N} \) acts
c) The point of contact between the acetabulum and the femur where the reaction force \( \mathbf{R} \) acts
d) The point where the tension in the hip abductor muscles \( \mathbf{T} \) acts
Answers: An important aspect of this problem, which is a part of a more involved problem, is that we want the tension in the hip abductor muscle, not the reaction force $\vec{R}$ (magnitude $R$ and angle $\beta$).

a) Using the center of the system is often a good idea, but in this case the leg is only one part of the larger system (the entire body). Specifically, we would need to include $\vec{R}$ in both the force and torque calculations. This can be done, of course, and a decided advantage would be that we obtain $\vec{R}$ from the resulting algebra.

b) Using the contact point with the ground would mean that the normal force $\vec{N}$ (and any friction force, not included in the problem but a great aid in standing on one foot) would not exert any torque. However, as in (a), $\vec{R}$ is then part of the problem.

c) Since the stated goal is to find $T$, using this contact point for determination of torques means that $\vec{R}$ does not enter the torque calculation. If the weight $N = |\vec{N}|$ is known or given, then only the torque equation, not any force equations, is needed.

d) Using this point takes $\vec{T}$ out of the torque equation entirely, but $T$ is what we want, so this would involve extra calculations and determination of $\vec{R}$.

Problem 3: Person standing on a Hill

A person is standing on a hill that is sloped at an angle of $\alpha$ with respect to the horizontal. The person’s legs are separated by a distance $d$, with one foot uphill and one downhill. The center of mass of the person is at a distance $h$ above the ground, perpendicular to the hillside, midway between the person’s feet. Assume that the coefficient of static friction between the person’s feet and the hill is sufficiently large that the person will not slip.

a) What is the magnitude of the normal force on each foot?
b) How far must the feet be apart so that the normal force on the upper foot is just zero? This is the moment when the person starts to rotate and fall over.

Solution:

The force diagram on the person is shown in the figure below. Note that the contact forces have been decomposed into components perpendicular and parallel to the hillside. A choice of unit vectors and positive direction for torque is also shown.

Applying Newton’s Second Law to the two components of the net force,

\[ \hat{j}: \quad N_1 + N_2 - mg \cos \alpha = 0 \]  
\[ \hat{i}: \quad f_1 + f_2 - mg \sin \alpha = 0. \]

These two equations imply that

\[ N_1 + N_2 = mg \cos \alpha \]  
\[ f_1 + f_2 = mg \sin \alpha. \]

Evaluating torques about the center of mass,

\[ h(f_1 + f_2) + (N_2 - N_1) \frac{d}{2} = 0. \]

Equation (3.5) can be rewritten as
\[ N_1 - N_2 = \frac{2h(f_1 + f_2)}{d}. \] \hspace{1cm} (3.6)

Substitution of Equation (3.4) into Equation (3.6) yields

\[ N_1 - N_2 = \frac{2h(mg \sin \alpha)}{d}. \] \hspace{1cm} (3.7)

We can solve for \( N_1 \) by adding Equations (3.3) and (3.7) and dividing by 2, giving

\[ N_1 = \frac{1}{2} mg \cos \alpha + \frac{h(mg \sin \alpha)}{d} = mg \left( \frac{1}{2} \cos \alpha + \frac{h}{d} \sin \alpha \right). \] \hspace{1cm} (3.8)

Similarly, we can solve for \( N_2 \) by subtracting Equation (3.7) from Equation (3.3) and dividing by 2, giving

\[ N_2 = mg \left( \frac{1}{2} \cos \alpha - \frac{h}{d} \sin \alpha \right). \] \hspace{1cm} (3.9)

The normal force \( N_2 \) as given in Equation (3.9) vanishes when

\[ \frac{1}{2} \cos \alpha = \frac{h}{d} \sin \alpha, \] \hspace{1cm} (3.10)

which can be solved for the minimum distance between the legs,

\[ d = 2h(\tan \alpha). \] \hspace{1cm} (3.11)

In the above figures, \( \alpha = 20^\circ, \) \( 2\tan \alpha = 0.73 \) and the stick-figure person is very close to tipping over.

It should be noted that no specific model for the friction force was used, that is, no coefficient of static friction entered the problem. The two friction forces \( f_1 \) and \( f_2 \) were not determined separately; only their sum entered the above calculations.

**Problem 4: Static Equilibrium: Rope Between Trees**

Suppose a rope of mass \( m = 0.1 \text{kg} \) is connected at the same height to two walls and is allowed to hang under its own weight. At both contact points between the rope and the wall, the rope makes an angle \( \theta = 60^\circ \) with respect to the vertical. In order to find the
tension in the rope at the ends and at the middle of the rope, you will need to think cleverly about what to include as the system in your free body diagram.

![Diagram of a rope with tension at ends and midpoint](image)

a) What is the tension at the ends of the rope where they are connected to the wall? Include in your answer your free body force diagram. Show all the forces acting on the rope and your choice of unit vectors.

b) What is the tension in the rope at the point midway between the walls? Include in your answer your free body force diagram. Show all the forces acting on the rope and your choice of unit vectors.

**Solution:**

The key to this problem is in understanding how to choose a force diagram for an extended body. Note that we are trying to find the tension at the ends of the rope and at the midpoint. We defined tension at a point in a rope to be:

$$T(x) = \left| \mathbf{F}_{\text{left, right}}(x) \right| = \left| \mathbf{F}_{\text{right, left}}(x) \right|. \quad (4.1)$$

This definition suggests that we need to slice the rope at the midpoint to calculate the tension at the midpoint. Let’s consider then only half the rope. The forces on half the rope are the tension $\mathbf{T}_{\text{end}}$ at the end, the tension $\mathbf{T}_{\text{mid}}$ at the midpoint and the gravitational force between the half of the rope and the earth, $(m/2)\mathbf{g}$. The free body diagram for the left half of the rope is shown in the figure below, with a suitable choice of unit vectors.
Since the rope is in static equilibrium, the sum of the forces is zero. The sum of the components of the forces in the $\hat{j}$-direction is zero,

$$\hat{j}: |\mathbf{T}_{\text{end}}| \cos \theta - \frac{(m/2)g}{\cos \theta} = 0.$$ \hspace{1cm} (4.2)

Therefore the tension at the end is

$$|\mathbf{T}_{\text{end}}| = \frac{(m/2)g}{\cos \theta} = \frac{((0.1 \text{ kg})/2)(9.8 \text{ m} \cdot \text{s}^{-2})}{\cos 60^\circ} = 0.98 \text{ N}.$$ \hspace{1cm} (4.3)

The sum of the components of the forces in the $\hat{i}$-direction is zero.

$$\hat{i}: -|\mathbf{T}_{\text{end}}| \sin \theta + |\mathbf{T}_{\text{mid}}| = 0.$$ \hspace{1cm} (4.4)

Substitute Equation (4.3) into Equation (4.4), yielding

$$\frac{-(m/2)g \sin \theta}{\cos \theta} + |\mathbf{T}_{\text{mid}}| = 0.$$ \hspace{1cm} (4.5)

Then solve for tension in the middle of the rope,

$$|\mathbf{T}_{\text{mid}}| = \frac{(m/2)g \tan \theta = ((0.1 \text{ kg})/2)(9.8 \text{ m} \cdot \text{s}^{-2}) \tan 60^\circ = |\mathbf{T}_{\text{mid}}| \sin 60^\circ = 0.85 \text{ N}.}$$ \hspace{1cm} (4.6)
Problem 5: The Knee

A man of mass \( m = 70 \text{kg} \) is about to start a race. Assume the runner’s weight is equally distributed on both legs. The patellar ligament in the knee is attached to the upper tibia and runs over the kneecap. When the knee is bent, a tensile force, \( \mathbf{T} \), that the ligament exerts on the upper tibia, is directed at an angle of \( \theta = 40^\circ \) with respect to the horizontal. The femur exerts a force \( \mathbf{F} \) on the upper tibia. The angle, \( \alpha \), that this force makes with the vertical will vary and is one of the unknowns to solve for. Assume that the ligament is connected a distance, \( d = 3.8 \text{cm} \), directly below the contact point of the femur on the tibia. The contact point between the foot and the ground is a distance \( s = 3.6 \times 10^1 \text{cm} \) from the vertical line passing through contact point of the femur on the tibia. The center of mass of the lower leg lies a distance \( x = 1.8 \times 10^1 \text{cm} \) from this same vertical line. Suppose the mass \( m_L \) of the lower leg is a 1/10 of the mass of the body.

\( a \) Find the magnitude \( T \) of the force \( \mathbf{T} \) of the patellar ligament on the tibia.

\( b \) Find the direction (the angle \( \alpha \)) of the force \( \mathbf{F} \) of the femur on the tibia.

\( c \) Find the magnitude \( F \) of the force \( \mathbf{F} \) of the femur on the tibia.

For a more detailed picture of this “articulation,” see Illustrations. Fig. 345. Gray, Henry. 1918. Anatomy of the Human Body.
**Solutions:**

a) Choose the unit vector \( \hat{i} \) to be directed horizontally to the right and \( \hat{j} \) directed vertically upwards. The two conditions for static equilibrium are

1. The sum of the forces acting on the rigid body is zero,
   \[
   \mathbf{F}_{\text{total}} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 + \mathbf{F}_4 = \mathbf{0}. 
   \]
   (5.1)

2. The vector sum of the torques about any point \( S \) in a rigid body is zero,
   \[
   \mathbf{\tau}_{S,\text{total}} = \mathbf{\tau}_{S,1} + \mathbf{\tau}_{S,2} + \mathbf{\tau}_{S,3} = \mathbf{0}. 
   \]
   (5.2)

The first condition that the sum of the forces is zero becomes

\[
\begin{align*}
\hat{i}: & -F \sin \alpha + T \cos \theta = 0 \\
\hat{j}: & N - F \cos \alpha + T \sin \theta - (1/10)mg = 0.
\end{align*}
\]
(5.3)

Since the weight is evenly distributed on the two feet, the normal force on one foot is equal to half the weight, or

\[
N = (1/2)mg; 
\]
(5.4)

the second equation in (5.3) becomes

\[
\begin{align*}
\hat{j}: & (1/2)mg - F \cos \alpha + T \sin \theta - (1/10)mg = 0 \\
& (2/5)mg - F \cos \alpha + T \sin \theta = 0.
\end{align*}
\]
(5.5)

The torque-force diagram on the knee is shown below.
Choose the point of action of the ligament on the tibia as the point $S$ about which to compute torques. Note that the tensile force, $T$, that the ligament exerts on the upper tibia will make no contribution to the torque about this point $S$. This may help slightly in doing the calculations. Choose counterclockwise as the positive direction for the torque; this is the positive $\hat{k}$-direction.

Then the torque due to the force $\vec{F}$ of the femur on the tibia is

$$\hat{r}_{s,1} = \hat{r}_{s,1} \times \vec{F} = d \hat{j} \times (-F \sin \alpha \hat{i} - F \cos \alpha \hat{j}) = d F \sin \alpha \hat{k}.$$  \hspace{1cm} (5.6)

The torque due to the mass of the leg is

$$\hat{r}_{s,2} = \hat{r}_{s,2} \times (-mg/10) = (-x \hat{i} - y_L \hat{j}) \times (-mg/10) \hat{j} = (1/10)x mg \hat{k}.$$ \hspace{1cm} (5.7)

The torque due to the normal force of the ground is

$$\hat{r}_{s,3} = \hat{r}_{s,3} \times N \hat{j} = (-s \hat{i} - y_N \hat{j}) \times N \hat{j} = -s N \hat{k} = -(1/2)s mg \hat{k}. \hspace{1cm} (5.8)$$

(In Equations (5.7) and (5.8), $y_L$ and $y_N$ are the vertical displacements of the point where the weight of the leg and the normal force with respect to the point $S$; as can be seen, these quantities do not enter directly into the calculations.)

The condition that the total torque about the point $S$ vanishes,
\[ \hat{\tau}_{S,\text{total}} = \hat{\tau}_{S,1} + \hat{\tau}_{S,2} + \hat{\tau}_{S,3} = \hat{0}, \]  

(5.9)

then becomes

\[ d F \sin \alpha \hat{k} + (1/10)xm \hat{k} - (1/2)s mg \hat{k} = \hat{0}. \]  

(5.10)

The torque equation to be used is then

\[ d F \sin \alpha + (1/10)x mg - (1/2)s mg = 0 \]  

(5.11)

The three equations in the three unknowns are summarized below:

\[ -F \sin \alpha + T \cos \theta = 0 \]
\[ (2/5)mg - F \cos \alpha + T \sin \theta = 0 \]  

(5.12)
\[ d F \sin \alpha + (1/10)x mg - (1/2)s mg = 0. \]

The horizontal force equation, the first in (5.12), implies that

\[ F \sin \alpha = T \cos \theta. \]  

(5.13)

Substituting this into the torque equation, the third of (5.12), yields

\[ d T \cos \theta + (1/10)x mg - s(1/2) mg = 0. \]  

(5.14)

It is essential that you understand that Equation (5.14) is the equation that would have been obtained if we had chosen the contact point between the tibia and the femur as the point about which to determine torques. Had we chosen this point, we would have saved one minor algebraic step.

We can solve this Equation (5.14) for the magnitude \( T \) of the force \( \hat{T} \) of the patellar ligament on the tibia,

\[ T = \frac{s(1/2)mg - (1/10)x mg}{d \cos \theta}. \]  

(5.15)

Inserting numerical values into Equation (5.15),

\[ T = (70\text{kg})(9.8\text{m} \cdot \text{s}^{-2})\left(3.6 \times 10^{-1}\text{m}\right)(1/2) - (1/10)(1.8 \times 10^{-1}\text{m}) \]
\[ \frac{3.8 \times 10^{-2}\text{m}}{\cos(40^\circ)} \]  

(5.16)

\[ = 3.8 \times 10^3 \text{N}. \]
b) We can now solve for the direction $\alpha$ of the force $\vec{F}$ of the femur on the tibia as follows. Rewrite the two force equations in (5.12) as

$$F \cos \alpha = (2/5)mg + T \sin \theta$$
$$F \sin \alpha = T \cos \theta.$$  

(5.17)

Dividing these equations yields

$$\frac{F \cos \alpha}{F \sin \alpha} = \cot \alpha = \frac{(2/5)mg + T \sin \theta}{T \cos \theta},$$

(5.18)

And so

$$\alpha = \cot^{-1} \left( \frac{(2/5)mg + T \sin \theta}{T \cos \theta} \right).$$

(5.19)

Thus,

$$\alpha = \cot^{-1} \left( \frac{(2/5)(70 \text{ kg})(9.8 \text{ m/s}^2)(3.4 \times 10^3 \text{ N}) \sin(40^\circ)}{(3.4 \times 10^3 \text{ N}) \cos(40^\circ)} \right) = 47^\circ.$$  

(5.20)

c) We can now use the horizontal force equation to calculate the magnitude $F$ of the force of the femur $\vec{F}$ on the tibia from Equation (5.13),

$$F = \frac{(3.8 \times 10^3 \text{ N}) \cos(40^\circ)}{\sin(47^\circ)} = 4.0 \times 10^3 \text{ N}.$$  

(5.20)

**Extra:**

The algebra in this problem was tedious but not difficult. Such problems are what computers are for. The following is a set of simple (not the most efficient, but the most direct) MAPLE commands for solving the three equations in (5.12) simultaneously.

```
> eq1:=-F*sin(alpha)+T*cos(theta)=0;
> eq2:=m*g/2-F*cos(alpha)+T*sin(theta)-m*g/10=0;
> eq3:=d*F*sin(alpha)+x*m*g/10-s*m*g/2=0;
> theta:=40*Pi/180; m:=70; g:=9.8; s:=.36; x:=.18; d:=.038;
> solve({eq1,eq2,eq3},{F,T,alpha});
```

Note that the angle $\alpha$ was converted from degrees to radians “by hand.” If you run these commands as given, you will obtain two sets of solutions, one with $\alpha$ in the first quadrant and the other with $\pi - \alpha$. In retrospect, this makes sense; if in each equation in
(5.12) we change the sign of $F$ and both trig functions of $\alpha$, we get the same equations back. So, pick the solution you want, but you should convert $\alpha$ back to degrees.

If this were an 8.012-type problem, we might want to find a symbolic expression for $\alpha$ that did not involve the intermediate numerical calculation of the tension. This is rather complicated algebraically; basically, the last two equations in (5.12) are solved for $F$ and $T$ in terms of $\alpha$, $\theta$ and the other variables (Cramer’s Rule is suggested) and the results substituted into the first of (5.12). The resulting expression is

$$\cot \alpha = \frac{(s/2 - x/10)\sin 40^\circ + (2d/5)\cos 40^\circ}{(s/2 - x/10)\cos 40^\circ}$$

$$= \tan 40^\circ + \frac{2d/5}{s/2 - x/10}$$

(5.21)

which leads to the same numerical result, $\alpha = 47^\circ$. 