Math Review Night
Vector Product, Angular Momentum, and Torque
Summary: Cross Product

Magnitude: equal to the area of the parallelogram defined by the two vectors

$$|\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta = |\vec{A}||\vec{B}| \sin \theta = (|\vec{A}| \sin \theta)|\vec{B}| \quad (0 \leq \theta \leq \pi)$$

Direction: determined by the Right-Hand-Rule
Properties of Cross Products

\[ \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \]

\[ c(\vec{A} \times \vec{B}) = \vec{A} \times c\vec{B} = c\vec{A} \times \vec{B} \]

\[ (\vec{A} + \vec{B}) \times \vec{C} = \vec{A} \times \vec{C} + \vec{B} \times \vec{C} \]
Cross Product of Unit Vectors

- Unit vectors in Cartesian coordinates

\[ \hat{i} \times \hat{j} = \hat{k} \quad \hat{i} \times \hat{i} = \vec{0} \]
\[ \hat{j} \times \hat{k} = \hat{i} \quad \hat{j} \times \hat{j} = \vec{0} \]
\[ \hat{k} \times \hat{i} = \hat{j} \quad \hat{k} \times \hat{k} = \vec{0} \]

\[ |\hat{i} \times \hat{j}| = |\hat{i}| \| \hat{j} \| \sin(\pi/2) = 1 \]
\[ |\hat{i} \times \hat{i}| = |\hat{i}| \| \hat{j} \| \sin(0) = 0 \]
Components of Cross Product

\[ \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k}, \quad \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \]

\[ \vec{A} \times \vec{B} = (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k} \]

\[
\begin{vmatrix}
\hat{i} & \hat{j} & \hat{k} \\
A_x & A_y & A_z \\
B_x & B_y & B_z \\
\end{vmatrix}
\]
Find a unit vector perpendicular to

\[ \vec{A} = \hat{i} + \hat{j} - \hat{k} \]

and

\[ \vec{B} = -2\hat{i} - \hat{j} + 3\hat{k} \]
Cross Product of Unit Vectors

- Unit vectors in cylindrical coordinates

\[ \hat{r} \times \hat{\theta} = \hat{k} \]
\[ \hat{\theta} \times \hat{k} = \hat{r} \]
\[ \hat{k} \times \hat{r} = \hat{\theta} \]

\[ \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \]

\[ \hat{r} \times \hat{r} = \hat{\theta} \times \hat{\theta} = \hat{k} \times \hat{k} = \vec{0} \]
Angular Momentum of a Point Particle

- Point particle of mass \( m \) moving with a velocity \( \vec{V} \)

- Momentum \( \vec{p} = m\vec{v} \)

- Fix a point \( S \)

- Vector \( \vec{r}_S \) from the point \( S \) to the location of the object

- Angular momentum about the point \( S \)

\[ \vec{L}_S = \vec{r}_S \times \vec{p} \]

- SI Unit \( [kg \cdot m^2 \cdot s^{-1}] \)
Cross Product: Angular Momentum of a Point Particle

\[ \vec{L}_S = \vec{r}_S \times \vec{p} \]

Magnitude:

\[ |\vec{L}_S| = |\vec{r}_S| |\vec{p}| \sin \theta \]

a) moment arm

\[ r_{S,\perp} = |\vec{r}_S| \sin \theta \]

b) Perpendicular momentum

\[ |\vec{L}_S| = r_{S,\perp} |\vec{p}| \]

\[ p_{S,\perp} = |\vec{p}| \sin \theta \quad |\vec{L}_S| = |\vec{r}_S| p_{\perp} \]
Angular Momentum of a Point Particle: Direction

Direction: Right Hand Rule
Worked Example: Angular Momentum and Cross Product

A particle of mass \( m = 2 \text{ kg} \) moves with a uniform velocity

\[
\vec{v} = 3.0 \text{ m} \cdot \text{s}^{-1} \hat{i} + 3.0 \text{ m} \cdot \text{s}^{-1} \hat{j}
\]

At time \( t \), the position vector of the particle with respect to the point \( S \) is

\[
\vec{r}_S = 2.0 \text{ m} \hat{i} + 3.0 \text{ m} \hat{j}
\]

Find the direction and the magnitude of the angular momentum about the origin, (the point \( S \)) at time \( t \).
Solution: Angular Momentum and Cross Product

The angular momentum vector of the particle about the point $S$ is given by:

$$\vec{L}_S = \vec{r}_S \times \vec{p} = \vec{r}_S \times m \vec{v}$$

$$= (2.0 \text{ m } \hat{i} + 3.0 \text{ m } \hat{j}) \times (2 \text{ kg})(3.0 \text{ m } \text{s}^{-1} \hat{i} + 3.0 \text{ m } \text{s}^{-1} \hat{j})$$

$$= 12 \text{ kg } \text{m}^2 \text{s}^{-1} \hat{k} + 18 \text{ kg } \text{m}^2 \text{s}^{-1}(-\hat{k})$$

$$= -6.0 \text{ kg } \text{m}^2 \text{s}^{-1} \hat{k}.$$ 

The direction is in the negative $\hat{k}$ direction, and the magnitude is

$$|\vec{L}_S| = 6.0 \text{ kg } \text{m}^2 \text{s}^{-1}.$$

$$\vec{i} \times \vec{j} = \hat{k},$$

$$\vec{j} \times \vec{i} = -\hat{k},$$

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = 0.$$
Angular Momentum and Circular Motion of a Point Particle:

Fixed axis of rotation: \( z \)-axis

Angular velocity \( \vec{\omega} = \omega \hat{k} \)

Velocity
\[
\vec{v} = \vec{\omega} \times \vec{r} = \omega \hat{k} \times R \hat{r} = R\omega \hat{\theta}
\]

Angular momentum about the point \( S \)
\[
\vec{L}_S = \vec{r}_S \times \vec{p} = \vec{r}_S \times m\vec{v} = Rmv \hat{k} = RmR\omega \hat{k} = mR^2 \omega \hat{k}
\]
A dumbbell is rotating at a constant angular speed about its center (point A). How does the angular momentum about the point B compared to the angular momentum about point A, (as shown in the figure)?
A particle of mass $m$ moves in a circle of radius $R$ at an angular speed $\omega$ about the $z$ axis in a plane parallel to but a distance $h$ above the $x$-$y$ plane.

a) Find the magnitude and the direction of the angular momentum $\vec{L}_0$ relative to the origin.

b) Is this angular momentum relative to the origin constant? If yes, why? If no, why is it not constant?
Torque as a Vector

Force $\vec{F}_P$ exerted at a point P on a rigid body.
Vector $\vec{r}_{S,P}$ from a point S to the point P.

Torque about point S due to the force exerted at point P:

$$\vec{\tau}_S = \vec{r}_{S,P} \times \vec{F}_P$$
Torque: Magnitude and Direction

Magnitude of torque about a point \( S \): 

\[
\tau_S = rF_\perp = rF \sin \theta
\]

where \( F \) is the magnitude of the force \( \vec{F}_P \).

Direction of torque:
Perpendicular to the plane formed by \( \vec{F}_P \) and \( \vec{r}_{S,P} \).
Determined by the Right-Hand-rule.
Consider two vectors \( \vec{r} = x\hat{i} \) with \( x > 0 \) and \( \vec{F} = F_x\hat{i} + F_z\hat{k} \) with \( F_x > 0 \) and \( F_z > 0 \). What is the direction of the cross product \( \vec{r} \times \vec{F} \)?
Conditions for Static Equilibrium

(1) Translational equilibrium: the sum of the forces acting on the rigid body is zero.

\[
\vec{F}_{\text{total}} = \vec{F}_1 + \vec{F}_2 + ... = \vec{0}
\]

(2) Rotational Equilibrium: the vector sum of the torques about any point S in a rigid body is zero.

\[
\tau_S^{\text{total}} = \tau_{S,1} + \tau_{S,2} + ... = \vec{0}
\]
Worked Example: Lever Law

Pivoted Lever at Center of Mass in Equilibrium

Show that (Lever Law):

\[ d_1 m_1 g = d_2 m_2 g \]
Worked Example: Pivoted Lever

Apply Newton’s 2\textsuperscript{nd} law to each body:

\[ F_{\text{pivot}} - m_Bg - N_{B,1} - N_{B,2} = 0 \]

\[ N_{1,B} - m_1g = 0 \quad N_{2,B} - m_2g = 0 \]

Third Law: \( N_{1,B} = N_{B,1} \quad N_{2,B} = N_{B,2} \)

Pivot force: \( F_{\text{pivot}} = (m_B + m_1 + m_2)g \)

Torque about pivot: \( d_1N_1\hat{k} - d_2N_2\hat{k} = \vec{0} \)

Lever Law: \( d_1m_1g = d_2m_2g \)
Generalized Lever Law

\[ \vec{F}_1 = \vec{F}_1,|| + \vec{F}_1,\perp \]
\[ \vec{F}_2 = \vec{F}_2,|| + \vec{F}_2,\perp \]

\[ F_{1,\perp} = F_1 \sin \theta_1 \]
\[ F_{2,\perp} = F_2 \sin \theta_2 \]

\[ \vec{\tau}_{\text{total}}^{\text{cm}} = \vec{\tau}_{\text{cm},1} + \vec{\tau}_{\text{cm},2} = \vec{0} \quad \rightarrow \quad (d_1 F_1,\perp - d_2 F_2,\perp) \hat{k} = 0 \]

\[ d_1 F_{1,\perp} = d_2 F_{2,\perp} \]
Problem Solving Strategy

**Force:**

1. Identify System and draw all forces and where they act on Free Body Force Diagram
2. Write down equations for static equilibrium of the forces: sum of forces is zero

**Torque:**

1. Choose point to analyze the torque about.
2. Choose sign convention for torque
3. Calculate torque about that point for each force. (Note sign of torque.)
4. Write down equation corresponding to condition for static equilibrium: sum of torques is zero
Checkpoint Problem: Lever Law

Suppose a beam of length $s = 1.0$ m and mass $m = 2.0$ kg is balanced on a pivot point that is placed directly beneath the center of the beam. Suppose a mass $m_1 = 0.3$ kg is placed a distance $d_1 = 0.4$ m to the right of the pivot point. A second mass $m_2 = 0.6$ kg is placed a distance $d_2$ to the left of the pivot point to keep the beam static.

(1) What is the force that the pivot exerts on the beam?

(2) What is the distance $d_2$ that maintains static equilibrium?
Problem: Ankle

A person of mass $m$ is crouching with their weight evenly distributed on both tiptoes. The force on the skeletal part of the foot are shown in the diagram. The normal force $N$ acts at the contact point between the foot and the ground. In this position, the tibia acts on the foot at the point $S$ with a force of magnitude $F$ and makes an angle $\beta$ with the vertical. This force acts on the ankle a horizontal distance $s$ from the point where the foot contacts the floor. The Achilles tendon also acts on the foot and is under considerable tension with magnitude $T$ and acts at an angle $\alpha$ with the horizontal as shown in the figure. The tendon acts on the ankle a horizontal distance $b$ from the point where the tibia acts on the foot. You may ignore the weight of the foot. Let $g$ be the gravitational constant. Compute the torque about the point $S$ due to

a) the normal force of the floor on the foot;
b) the tendon force on the foot;
c) the force of the tibia on the foot.