Math Review Night
Continuous Mass Flow
8.01
Category 1: Adding Rain

There is a transfer of material into the object but no transfer of momentum in the direction of motion of the object. Consider for example rain falling vertically downward into a moving cart. A small amount of rain has no component of momentum in the direction of motion of the cart.
Category 2: Leaking Sand

The material continually leaves the object but it does not transport any momentum away from the object in the direction of motion of the object. For example, consider an ice skater gliding on ice holding a bag of sand that is leaking straight down with respect to the moving skater.
Category 3: Impulse

The material continually hits the object providing an impulse resulting in a transfer of momentum to the object in the direction of motion. For example, suppose a fire hose is used to put out a fire on a boat. The incoming water continually hits the boat impulsing it forward.
Category 4: Recoil

The material continually is ejected from the object, resulting in a recoil of the object. For example when fuel is ejected from the back of a rocket, the rocket recoils forward.

\[ u \leftarrow \Delta m \rightarrow m \rightarrow \Delta v \]

Reference frame of rocket

Recoil forward speed increases by \( \Delta v \)

\[ u = \text{speed of ejected fuel relative to rocket} \]

Reference frame in which rocket moves with speed \( v \)

\[ \Delta m \rightarrow v - u \rightarrow v + \Delta v \]
Worked Example: Coal Car
(constant force, varying speed)

An empty coal car of mass \( m \) starts from rest under an applied force of magnitude \( F \). At the same time coal begins to run into the car at a steady rate \( b \) from a coal hopper at rest along the track. Find the speed when a mass \( m_c \) of coal has been transferred.
Worked Example: Emptying a Freight Car

An freight car of mass $m_c$ contains a mass of sand $m_s$. At $t = 0$ a constant horizontal force of magnitude $F$ is applied in the direction of rolling and at the same time a port in the bottom is opened to let the sand flow out at the constant rate $b = \frac{dm_s}{dt}$. Find the speed of the freight car when all the sand is gone. Assume that the freight car is at rest at $t = 0$. 

![Freight Car with force F applied](image.png)
Table Problem: Filling a Cart

Material is blown into cart A from cart B at a rate of b kilograms per second. The material leaves the chute vertically downward, so that it has the same horizontal velocity $u$, as cart B. At the moment of interest, cart A has mass $m_A$ and velocity $v$. Find an expression for the rate of change of velocity, the instantaneous acceleration, $dv/dt$. 
A Faster Journey to Mars

A plasma rocket engine now in development could reduce the travel time to Mars by two-thirds.

Conventional Rockets

Current chemical rockets burn their fuel quickly and then send their payload to glide on a curved path to its destination. Spacecraft like the Mars Phoenix Lander, which was launched atop a conventional rocket in 2007, take about 295 days to reach Mars.

Prototype Vasimr Rocket

Currently being tested, this plasma rocket engine uses argon gas and electromagnetic energy to generate thrust.

PLASMA ROCKET

Vasimr, short for Variable Specific Impulse Magnetoplasma Rocket, would provide weak but continuous thrust during a Mars flight. A craft using Vasimr rockets would spend a month in orbit around the Earth gaining speed, then reach Mars in about 85 days by continually adjusting its thrust to follow a faster, more direct path.

How It Works

To create thrust, Vasimr turns argon into plasma, a superheated state of matter in which negatively charged free electrons and positively charged ions mix together.

1. Gas is fed into the first stage. The plasma source coupler collides electrons into argon atoms, releasing more electrons and creating positive argon ions.
2. Ions and electrons move in tight helical paths along magnetic field lines, which prevent them from escaping.
3. Electromagnetic waves at a specific frequency heat the ions, but not the electrons. This expands the ions' paths, accelerating them to very high energies.
4. The magnetic field lines expand as they leave the rocket. This acts like a nozzle, accelerating the ionized plasma and providing one pound of thrust.

Testing in Space

In 2013, a twin-engine Vasimr is scheduled to be attached to the International Space Station for testing. If successful, it could eventually provide gentle thrust for boosting the station's orbit.

Source: Ast Astra Rocket Company
A rocket at time $t$ is moving with speed $v_{r,0}$ in the positive $x$-direction in empty space. The rocket burns the fuel at a rate $\frac{dm_{f,\text{out}}}{dt} = b > 0$. The fuel is ejected backward with speed $u$ relative to the rocket.

a) What is the relationship between the time rate of change of exhaust mass $\frac{dm_f}{dt}$, and the time rate of change of rocket mass $\frac{dm_r}{dt}$?

b) Find an equation for the rate of change of the speed of the rocket in terms $m_r(t)$, $u$, and $\frac{dm_r}{dt}$ and solve for $v$.

c) Find the differential equation describing the motion of the rocket if it is in a constant gravitational field of magnitude $g$. 
Strategy: Rocket Problem

• Goal: Determine velocity of rocket as function of time as mass is continuously ejected at rate \( \frac{dm_f}{dt} \) with speed \( u \) relative to rocket.

• System: consider all elements that undergo momentum change: rocket and fuel

• Using Momentum flow diagram, apply

\[
\vec{F}_{ext} = \lim_{\Delta t \to 0} \frac{\vec{P}_{\text{total}}(t + \Delta t) - \vec{P}_{\text{total}}(t)}{\Delta t}
\]

to find differential equation that describes motion.
Rocket Problem:

A rocket at time $t = 0$ is moving with speed $v_{r,0}$ in the positive $x$-direction in empty space. The rocket burns the fuel at a rate $\frac{dm_f}{dt} = b > 0$. The fuel is ejected backward with speed $u$ relative to the rocket. The goal is to find an equation for the rate of change of the speed of the rocket in terms $m_r(t), u,$ and $\frac{dm_r}{dt}$ and solve for $v$. 
1. Rocket with total mass $m_r(t)$ moves with speed $v_r(t)$ in positive $x$-direction according to observer

2. Total mass consists of mass of rocket $m_{r,0}$ and fuel $m_f(t)$

3. Fuel element with mass $\Delta m_f$, moves with speed of rocket $v_r(t)$ at time $t$, is ejected during interval $[t,t+\Delta t]$

4. $x$-component of momentum at time $t$

$$P_{\text{sys}}(t) = (m_r(t) + \Delta m_f)v_r(t)$$
State at $t + \Delta t$

- Rocket is propelled forward by ejected fuel with new rocket speed
  
  $$v_r(t + \Delta t) = v_r(t) + \Delta v_r$$

- Fuel is ejected backward with speed $u$ relative to rocket. Relative to observer’s frame, ejected fuel element has speed
  
  $$v_r + \Delta v_r - u$$

- $x$-component of system’s momentum at time $t+\Delta t$
  
  $$P_{x,\text{sys}}(t + \Delta t) = m_r(t)(v_r + \Delta v_r) + \Delta m_f(v_r + \Delta v_r - u)$$
Rocket Equation

Are there any external forces at time $t$?

Two cases:
1. Taking off: $F_{\text{ext}}(t) = m_r(t)\vec{g}$
2. Negligible gravitational field: $F_{\text{ext}} = 0$

Apply Momentum Principle:

$$F_{\text{ext}} = \lim_{\Delta t \to 0} \frac{\delta \vec{p}^{\text{total}}(t + \Delta t) - \delta \vec{p}^{\text{total}}(t)}{\Delta t}$$

$$F_{\text{ext}} = \lim_{\Delta t \to 0} \frac{m_r(t)(\vec{v}_r + \Delta \vec{v}_r) + \Delta m_f \vec{u} - m_r(t)\vec{v}_r}{\Delta t} = \lim_{\Delta t \to 0} \frac{m_r(t)\Delta \vec{v}_r + \Delta m_f \vec{u}}{\Delta t}$$

$$m_r(t)\vec{g} = m_r(t) \frac{d\vec{v}_r}{dt} + \frac{dm_f}{dt} \vec{u}$$

Conservation of mass: Rate of decrease of mass of rocket equals rate of ejection of mass

$$\frac{dm_f}{dt} = -\frac{dm_r(t)}{dt}$$

Rocket equation:

$$F_{\text{ext}} = m_r(t) \frac{d\vec{v}_r}{dt} - \frac{dm_r}{dt} \vec{u}$$
Rocket Equation in Gravitational Field

\[ \vec{F}_{\text{ext}} + \frac{dm_r}{dt} \vec{u} = m_r(t) \frac{d\vec{v}_r}{dt} \]

- Fuel ejection term can be interpreted as thrust force
- Relative fuel ejection velocity \( \vec{u} = -u\hat{k} \)
- External force \( \vec{F}_{\text{ext}}(t) = -m_cg\hat{k} \)
- Rocket equation

\[ -m_r g = m_r \frac{dv_{r,z}}{dt} + \frac{dm_r}{dt} u \]

\[ \frac{dv_{r,z}}{dt} = - \frac{1}{m_r} \frac{dm_r}{dt} u - g \]

- Integrate with respect to time

\[ \int_{t_0}^{t_f} \frac{dv_{r,z}}{dt} dt = - \int_{t_0}^{t_f} \frac{1}{m_r} \frac{dm_r}{dt} u dt - \int_{t_0}^{t_f} g dt \]

- Solution:

\[ v_{r,z}(t_f) = u \ln \frac{m_r(t = 0)}{m_r(t_f)} - gt_f \]
Rocket Equation in Gravitational Field

- Fuel ejection term can be interpreted as thrust force

\[ \vec{F}_{\text{ext}} + \frac{dm_r}{dt} \ddot{u} = m_r(t) \frac{d\vec{v}_r}{dt} \]

\[ \vec{F}_{\text{thrust}} = \frac{dm_r}{dt} \ddot{u} \]

- Solution: shorter the burn time, the greater the velocity

\[ v_{r,z}(t_f) = u \ln \frac{m_r(t = 0)}{m_r(t_f)} - gt_f \]
Concept Question

Suppose rain falls vertically into an open cart rolling along a straight horizontal track with negligible friction. As a result of the accumulating water, the speed of the cart

1. increases.
2. does not change.
3. decreases.
4. not sure.
5. not enough information is given to decide.
Concept Question

If a rocket in gravity-free outer space has the same thrust at all times, is its acceleration
1. constant?
2. increasing?
3. decreasing?
Concept Question: Rocket Fuel Burn Time

When a rocket accelerates in a gravitational field, will it reach a greater final velocity if the fuel burn time is

1. as fast as possible?

3. as slow as possible?

5. The final speed is independent of the fuel burn time?

4. I’m not sure.