Math Review: Rigid Body: Translation and Rotational Motion of a Rigid Body

1.1 Introduction

The general motion of a rigid body of mass $m$ consists of a translation of the center of mass with velocity $\vec{V}_{cm}$ and a rotation about the center of mass with all elements of the rigid body rotating with the same angular velocity $\vec{\omega}_{cm}$. We prove this result in Appendix 1.A. Figure 1.1 shows the center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

![Figure 1.1](image)

Figure 1.1 The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.

We shall begin by recalling our definition of the center of mass reference frame

1.2 Center of Mass Reference Frame

Consider a system of $N$ particles of total mass $m_{\text{total}}$, the $i^{th}$ particle in reference $O$ is located at position $\vec{r}_i$ with velocity $\vec{v}_i$. The momentum of the system is momentum $\vec{p}_{\text{sys}} = \sum_i m_i \vec{v}_i$. Recall that the vector from the origin of frame $O$ to the center of mass of the system of particles, is defined as

$$\vec{R}_{cm} = \frac{1}{m_{\text{total}}} \sum_{i=1}^{N} m_i \vec{r}_i.$$  \hfill (1.2.1)

The velocity of the center of mass in reference frame $O$ is given by
Define the center of mass reference frame \( O_{cm} \) as a reference frame moving with velocity \( \vec{V}_{cm} \) with respect to \( O \). Denote the position vector of the \( i^{th} \) particle with respect to origin of reference frame \( O \) by \( \vec{r}_i \) and similarly, denote the position vector of the \( i^{th} \) particle with respect to origin of reference frame \( O_{cm} \) by \( \vec{r}_{cm,i} \) (Figure 1.2).

\[
\vec{V}_{cm} = \frac{1}{m_{\text{total}}} \sum_i m_i \vec{v}_i = \frac{\vec{p}_{\text{sys}}}{m_{\text{total}}} \tag{1.2.2}
\]

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.2}
\caption{Position vector of \( i^{th} \) particle in the center of mass reference frame.}
\end{figure}

The position vector of the \( i^{th} \) particle in the two center of mass frame is then given by

\[
\vec{r}_{cm,i} = \vec{r}_i - \vec{R}_{cm} \tag{1.2.3}
\]

or equivalently the position of the \( i^{th} \) particle in the reference frame \( O \) can be expressed as

\[
\vec{r}_i = \vec{r}_{cm,i} + \vec{R}_{cm} \tag{1.2.4}
\]

The displacement of the \( i^{th} \) particle in the center of mass reference frame is then given by

\[
d\vec{r}_{cm,i} = d\vec{r}_i - d\vec{R}_{cm} \tag{1.2.5}
\]

The velocity of the \( i^{th} \) particle in the center of mass reference frame is then given by

\[
\vec{v}_{cm,i} = \vec{v}_i - \vec{V}_{cm} \tag{1.2.6}
\]

or equivalently the velocity of the \( i^{th} \) particle in the reference frame \( O \) can be expressed as

\[
\vec{v}_i = \vec{v}_{cm,i} + \vec{V}_{cm} \tag{1.2.7}
\]
1.3 Constrained Motion: Translation and Rotation

We shall encounter many examples of a rolling object whose motion is constrained. For example we will study the motion of an object rolling along a level or inclined surface and the motion of a yo-yo unwinding and winding along a string. We will examine the constraint conditions between the translational quantities that describe the motion of the center of mass, displacement, velocity and acceleration, and the rotational quantities that describe the motion about the center of mass, angular displacement, angular velocity and angular acceleration. We begin with a discussion about the rotation and translation of a rolling wheel (Figure 1.3).

Figure 1.3 rotating and translating wheel

Consider a wheel of radius $R$ is rolling in a straight line. The center of mass of the wheel is moving in a straight line at a constant velocity $\vec{V}_{cm}$. Let’s analyze the motion of a point $P$ on the rim of the wheel.

In a reference frame $O$ at rest with respect to the ground, let $\vec{v}_p$ denote the velocity of a point $P$ on the rim. In the center of mass reference frame $O_{cm}$ moving with velocity $\vec{V}_{cm}$ with respect to at $O$, let $\vec{v}_{cm,p}$ denote the velocity of the point $P$ on the rim. These three velocities are related by the law of addition of velocities (Eq. (1.2.7))

$$\vec{v}_p = \vec{v}_{cm,p} + \vec{V}_{cm}.$$  \hspace{1cm} (1.2.8)

Let’s choose Cartesian coordinates for the translation motion and polar coordinates for the motion about the center of mass as shown in the figures below.
The center of mass velocity in the reference frame fixed to the ground (Figure 1.4a) is given by

\[ \vec{V}_{cm} = V_{cm} \hat{i}. \]  

(1.2.9)

The position of the center of mass in the reference frame fixed to the ground is given by

\[ \vec{R}_{cm}(t) = (X_{cm0} + V_{cm}t) \hat{i} + R \hat{j} \]  

(1.2.10)

where \( X_{cm0} \) is the initial x-component of the center of mass at \( t = 0 \).

The point \( P \) on the rim is undergoing uniform circular motion with the velocity in the center of mass reference frame (Figure 1.4b) given by

\[ \vec{v}_{cm,P} = R \omega_{cm} \hat{\theta} \]  

(1.2.11)

If we want to use the law of addition of velocities then we should express \( \vec{v}_{cm,P} = R \omega_{cm} \hat{\theta} \) in Cartesian coordinates. Assume that at \( t = 0 \), \( \theta(t = 0) = 0 \) i.e. the point \( P \) is at the top of the wheel at \( t = 0 \). From the Figure 1.5 we see that

\[ \hat{r} = \sin \theta \hat{i} + \cos \theta \hat{j} \]

\[ \hat{\theta} = \cos \theta \hat{i} - \sin \theta \hat{j} \]  

(1.2.12)

Therefore the velocity of the point \( P \) on the rim in the center of mass reference frame is given by

\[ \vec{v}_{cm,P} = R \omega_{cm} \hat{\theta} = R \omega_{cm} (\cos \theta \hat{i} - \sin \theta \hat{j}) \]  

(1.2.13)
Figure 1.5 Vector decomposition of unit vectors in the center of mass frame

Now substitute Eqs. (1.2.9) and (1.2.13) into Eq. (1.2.8) for the velocity of a point $P$ on the rim in the reference frame fixed to the ground

$$
\ddot{v}_p = R\omega_{cm}(\cos\theta\hat{i} - \sin\theta\hat{j}) + \dot{V}_{cm}\hat{i} \\
= (V_{cm} + R\omega_{cm}\cos\theta)\hat{i} - R\omega_{cm}\sin\theta\hat{j}.
$$

(1.2.14)

The point $P$ is in contact with the ground when $\theta = \pi$. At that instant the velocity of a point $P$ on the rim in the reference frame fixed to the ground is

$$
\ddot{v}_p(\theta = \pi) = (V_{cm} - R\omega_{cm})\hat{i}.
$$

(1.2.15)

What velocity does the observer at rest on the ground measure for the point on the rim when that point is in contact with the ground? There are three cases when $\omega_{cm} \neq 0$,

We begin by assuming $\omega_{cm} > 0$. In order to understand the relationship between $V_{cm}$ and $\omega_{cm}$, we consider the displacement of the center of mass for a small time interval $\Delta t$. reference frames.
From Eq. (1.2.10)

\[ \Delta \mathbf{R}_{\text{cm}}(t) = \mathbf{R}_{\text{cm}}(t + \Delta t) - \mathbf{R}_{\text{cm}}(t) = \Delta X_{\text{cm}} \hat{i} = V_{\text{cm}} \Delta t \hat{i} \quad (1.2.16) \]

The point \( P \) on the rim in the center of mass reference frame is undergoing circular motion so the magnitude of the tangential displacement is given by the arc length subtended by an angular displacement \( \Delta \theta = \omega_{\text{cm}} \Delta t \),

\[ \Delta s = R \Delta \theta = R \omega_{\text{cm}} \Delta t \quad (1.2.17) \]

In Figure 1.6 we show the displacement of a point \( P \) on the rim in the two reference frames.

If the x-component of the displacement of the center of mass is greater than the arc length subtended by \( \Delta \theta \), then the wheel is \textit{skidding} along the surface

\[ \Delta X_{\text{cm}} > \Delta s \quad (1.2.18) \]

Using Eqs. (1.2.16) and (1.2.17), Eq. (1.2.18) becomes

\[ V_{\text{cm}} \Delta t > R \omega_{\text{cm}} \Delta t \quad (1.2.19) \]
So dividing through by $\Delta t$ in Eq. (1.2.19), the skidding condition becomes

$$
V_{cm} > R\omega_{cm}, \text{ skidding}
$$

(1.2.20)

If the x-component of the displacement of the center of mass is lesser than the arc length subtended by $\Delta \theta$, then the wheel is slipping along the surface

$$
\Delta X_{cm} < \Delta s
$$

(1.2.21)

Arguing as above the slipping condition becomes

$$
V_{cm} < R\omega_{cm}, \text{ slipping}
$$

(1.2.22)

If the x-component of the displacement of the center of mass is equal to the arc length subtended by $\Delta \theta$, then the wheel is rolling without slipping or skidding, \textit{rolling without slipping} for short, along the surface

$$
\Delta X_{cm} = \Delta s
$$

(1.2.23)

Again arguing as above, the rolling without slipping condition becomes

$$
V_{cm} = R\omega_{cm}, \text{ rolling without slipping}
$$

(1.2.24)

\textbf{Rolling without slipping: (}$V_{cm} = R\omega_{cm}$\textbf{)}

When a wheel is rolling without slipping, the velocity of a point $P$ on the rim when it is in contact with the ground ($\theta = \pi$) is zero (Eq. (1.2.15),

$$
\vec{v}_P(\theta = \pi) = (V_{cm} - R\omega_{cm})\hat{i} = (R\omega_{cm} - R\omega_{cm})\hat{i} = \vec{0}.
$$

(1.2.25)

This makes sense because the velocity of the point $P$ on the rim in the center of mass reference frame when it is in contact with the ground points in the opposite direction as the translational motion of the center of mass of the wheel. The two velocities have the same magnitude so the vector sum is zero. The observer at rest on the ground sees the contact point on the rim at rest relative to the ground.

Thus any friction force acting between the tire and the ground for a wheel that is rolling without slipping is static friction because the two surfaces are instantaneously at rest with respect to each other. Recall that the direction of the static friction force depends on the other forces acting on the wheel.
1.3.1 Example Bicycle wheel rolling without slipping

Consider a bicycle wheel of radius $R$ that is rolling in a straight line without slipping. The velocity of the center of mass in a reference frame fixed to the ground is given by velocity $\vec{V}_{cm}$. A bead is fixed to a spoke a distance $b$ from the center of the wheel.

a) Find the position, velocity, and acceleration of the bead as a function of time in the center of mass reference frame.

b) Find the position, velocity, and acceleration of the bead as a function of time as seen in a reference frame fixed to the ground.

Solution:

a) Choose a reference frame with an origin at the center of the wheel, and moving with the wheel. Choose polar coordinates. The angular velocity is $\omega_{cm} = \frac{d\theta}{dt}$.

Then the bead is moving uniformly in a circle of radius $r = b$ with the position, velocity, and acceleration given by

$$\vec{r}_{cm,b} = b \hat{r}, \quad \vec{v}_{cm,b} = b\omega_{cm} \hat{\theta}, \quad \vec{a}_{cm,b} = -b\omega_{cm}^2 \hat{r} \quad (1.2.26)$$

Because the wheel is rolling without slipping, the velocity of a point on the rim of the wheel has speed $v_{cm,r} = R\omega_{cm}$. This is equal to the speed of the center of mass of the wheel $V_{cm}$, thus

$$V_{cm} = R\omega_{cm} \quad (1.2.27)$$
Note that at $t = 0$, the angle $\theta = \theta_0 = 0$. So the angle grows in time as

$$\theta(t) = \omega_{cm} t = (V_{cm} / R) t$$  \hspace{1cm} (1.2.28)

So the velocity and acceleration of the bead with respect to the center of the wheel become

$$\mathbf{v}_{cm,b} = \frac{b V_{cm}}{R} \hat{\theta}, \quad \mathbf{a}_{cm,b} = -\frac{b V_{cm}^2}{R^2} \hat{r}$$  \hspace{1cm} (1.2.29)

b) Define a second reference frame fixed to the ground with choice of origin, Cartesian coordinates and unit vectors as shown in the figure below.

Then the position vector of the center of mass in the reference frame fixed to the ground is given by

$$\mathbf{R}_{cm}(t) = X_{cm} \hat{i} + R \hat{j} = V_{cm} t \hat{i} + R \hat{j}.$$  \hspace{1cm} (1.2.30)

The relative velocity of the two frames is the derivative

$$\mathbf{V}_c = \frac{d\mathbf{R}_c}{dt} = \frac{dX_{cm}}{dt} \hat{i} = V_{cm} \hat{i}.$$  \hspace{1cm} (1.2.31)

Since the center of the wheel is moving at a uniform speed the relative acceleration of the two frames is zero,

$$\mathbf{A}_c = \frac{d\mathbf{V}_c}{dt} = \hat{0}. $$  \hspace{1cm} (1.2.32)

Define the position, velocity, and acceleration in this frame (with respect to the ground) by

$$\mathbf{r}_b(t) = x_b(t) \hat{i} + y_b(t) \hat{j}, \quad \mathbf{v}_b(t) = v_{x,b}(t) \hat{i} + v_{y,b}(t) \hat{j}, \quad \mathbf{a}_b(t) = a_{x,b}(t) \hat{i} + a_{y,b}(t) \hat{j}.$$  \hspace{1cm} (1.2.33)
Then the position vectors are related by

$$\vec{r}_b(t) = \vec{R}_{cm}(t) + \vec{r}_{cm,b}(t).$$  \hspace{1cm} (1.2.34)

In order to add these vectors we need to decompose the position vector in the center of mass reference frame into Cartesian components,

$$\vec{r}_{cm,b}(t) = b \hat{r}(t) = b \sin(\theta(t)) \hat{i} + b \cos(\theta(t)) \hat{j}. \hspace{1cm} (1.2.35)$$

Then using the relation $\theta(t) = \left(V_{cm} / R\right) t$

$$\vec{r}_b(t) = \vec{R}_{cm}(t) + \vec{r}_{cm,b}(t) = \left(V_{cm} t \hat{i} + R \hat{j}\right) + \left(b \sin(\theta(t)) \hat{i} + b \cos(\theta(t)) \hat{j}\right)$$

$$= \left( V_{cm} t + b \sin \left(\frac{V_{cm}}{R} t\right) \right) \hat{i} + \left( R + b \cos \left(\frac{V_{cm}}{R} t\right) \right) \hat{j}. \hspace{1cm} (1.2.36)$$

Thus the position components of the bead with respect to the reference frame fixed to the ground are given by

$$x_b(t) = V_{cm} t + b \sin((V_{cm} / R)t) \hspace{1cm} (1.2.37)$$

$$y_b(t) = R + b \cos((V_{cm} / R)t). \hspace{1cm} (1.2.38)$$

A plot of the y-component vs. the x-component of the position of the bead in the reference frame fixed to the ground is shown below using the values $V_{cm} = 5 \text{ m} \cdot \text{s}^{-1}, R = 0.25 \text{ m}$, and $b = 0.125 \text{ m}$.

This path is called a cycloid. We can differentiate the position vector in the reference frame fixed to the ground to find the velocity of the bead.
\[ \ddot{\mathbf{r}}_b(t) = \frac{d\dot{\mathbf{r}}_b}{dt} = \frac{d}{dt}(V_{cm} t + b\sin((V_{cm} / R)t)) \ \hat{i} + \frac{d}{dt}(R + b\cos((V_{cm} / R)t)) \ \hat{j} \]  
\[ \ddot{\mathbf{v}}_b(t) = (V_{cm} + (b / R)V\cos((V_{cm} / R)t)) \ \hat{i} - ((b / R)V_{cm}\sin((V_{cm} / R)t)) \ \hat{j}. \]  

Alternatively, we can decompose the velocity of the bead in the center of mass reference frame into Cartesian coordinates

\[ \ddot{\mathbf{v}}_{cm,b}(t) = (b / R)V_{cm} \cos((V_{cm} / R)t) \ \hat{i} - \sin((V_{cm} / R)t) \ \hat{j}. \]  

Then the velocities are related by the law of addition of velocities

\[ \ddot{\mathbf{v}}_b(t) = \ddot{\mathbf{v}}_{cm} + \ddot{\mathbf{v}}_{cm,b}(t). \]  

so

\[ \ddot{\mathbf{v}}_b(t) = V_{cm} \ \hat{i} + (b / R)V_{cm} \cos((V_{cm} / R)t) \ \hat{i} - \sin((V_{cm} / R)t) \ \hat{j} \]  
\[ \ddot{\mathbf{v}}_b(t) = (V_{cm} + (b / R)V_{cm} \cos((V_{cm} / R)t)) \ \hat{i} - (b / R)\sin((V_{cm} / R)t) \ \hat{j} \]

in agreement with our previous result.

The acceleration is the same in either frame so

\[ \ddot{\mathbf{a}}_b(t) = \ddot{\mathbf{a}}_{cm,b} = -(b / R^2)V_{cm}^2 \ \hat{i} = -(b / R^2)V_{cm}^2 \sin((V_{cm} / R)t) \ \hat{i} + \cos((V_{cm} / R)t) \ \hat{j}. \]  

\[ \text{1.3.2 Example Cylinder rolling without slipping down an inclined plane} \]

A uniform cylinder of outer radius \( R \) and mass \( M \) with moment of inertia about the center of mass \( I_{cm} = (1/2)MR^2 \) starts from rest and moves down an incline tilted at an angle \( \beta \) from the horizontal. The center of mass of the cylinder has dropped a vertical distance \( h \) when it reaches the bottom of the incline. Let \( g \) denote the gravitational constant. The cylinder rolls without slipping down the incline. What is the relation between the component of the acceleration of the center of mass in the direction down the inclined plane and the component of the angular acceleration into the page of the figure shown below.

![Diagram of cylinder rolling down an inclined plane](image)
**Solution:** We begin by choosing coordinate systems for the translational and rotational motion as shown in the figure below.

For a time interval $\Delta t$, the displacement of the center of mass is given by

$$\Delta \mathbf{R}_{cm}(t) = \Delta \mathbf{X}_{cm} \hat{i}.$$  

The arc length due to the angular displacement of a point on the rim during the time interval $\Delta t$ is given by $\Delta s = R\Delta \theta$. The rolling without slipping condition is

$$\Delta X_{cm} = R \Delta \theta. $$

If we divide both sides by $\Delta t$ and take the limit as $\Delta t \to 0$ then the rolling without slipping condition show that the $x$-component of the center of mass velocity is equal to the magnitude of the tangential component of the velocity of a point on the rim

$$V_{cm} = \lim_{\Delta t \to 0} \frac{\Delta X_{cm}}{\Delta t} = \lim_{\Delta t \to 0} R \frac{\Delta \theta}{\Delta t} = R \omega_{cm}. $$

Similarly if we differentiate both sides of the above equation, we find a relation between the $x$-component of the center of mass acceleration is equal to the magnitude of the tangential component of the acceleration of a point on the rim

$$A_{cm} = \frac{dV_{cm}}{dt} = R \frac{d\omega_{cm}}{dt} = R \alpha_{cm}. $$

### 1.4 Translational Equation of Motion

We shall think about the system of particles as follows. We treat the whole system as a single point-like particle of mass $m_{total}$ located at the center of mass moving with the velocity of the center of mass $\mathbf{V}_{cm}$. The total external force acting on the system acts at the center of mass

$$\mathbf{F}_{ext}^{total} = \frac{d\mathbf{p}_{sys}^{total}}{dt} = \frac{d}{dt} (m_{total} \mathbf{V}_{cm}). \quad (1.3.1)$$
1.5 Angular Momentum for a System of Particles Undergoing Translational and Rotational

We shall now show that the total angular momentum of a body about a point \( S \) can be decomposed into two vector parts, the angular momentum of the center of mass motion about the point, and the angular momentum of the rotational motion about the center of mass.

Consider a system of \( N \) particles located at the points labeled \( i = 1, 2, ..., N \). The total angular momentum about the point \( S \) is

\[
\mathbf{L}_S^{\text{total}} = \sum_{i=1}^{N} \mathbf{L}_{S,i} = \left( \sum_{i=1}^{N} \mathbf{r}_{S,i} \times m_i \mathbf{v}_i \right),
\]

where \( \mathbf{r}_{S,i} \) is the vector from the point \( S \) to the \( i^{th} \) particle (See Figure 1.7).

![Figure 1.7 Position vector of \( i^{th} \) particle in the center of mass reference frame.](image)

If we choose the origin of reference frame \( O \) at the point \( S \), we can now substitute both Eqs. (1.2.4) and (1.2.7) into Eq. (1.4.1) noting that \( \mathbf{R}_{cm} = \mathbf{r}_{S,cm} \) yielding

\[
\mathbf{L}_S^{\text{total}} = \sum_{i=1}^{N} (\mathbf{r}_{cm,i} + \mathbf{R}_{cm}) \times m_i (\mathbf{v}_{cm,i} + \mathbf{v}_{cm}),
\]

(1.4.2)

When we expand the expression in Equation (1.4.2) we have four terms,

\[
\mathbf{L}_S^{\text{total}} = \sum_{i=1}^{N} (\mathbf{r}_{cm,i} \times m_i \mathbf{v}_{cm,i}) + \sum_{i=1}^{N} (\mathbf{R}_{cm} \times m_i \mathbf{v}_{cm}) + \sum_{i=1}^{N} (\mathbf{r}_{cm,i} \times m_i \mathbf{v}_{cm,i}) + \sum_{i=1}^{N} (\mathbf{r}_{cm,i} \times m_i \mathbf{v}_{cm})
\]

(1.4.3)
The vector \( \mathbf{R}_{cm} \) is a constant vector that depends only on the location of the center of mass and not on the location of the \( i \)th particle. Therefore in the first term in the above equation, \( \mathbf{R}_{cm} \) can be taken outside the summation. Similarly, in the second term the velocity of the center of mass \( \mathbf{V}_{cm} \) is the same for each term in the summation, and may be taken outside the summation.

\[
\mathbf{L}_{\text{tot}} = \mathbf{R}_{cm} \times \left( \sum_{i=1}^{N} m_i \mathbf{v}_{cm,i} \right) + \mathbf{R}_{cm} \times \left( \sum_{i=1}^{N} m_i \mathbf{v}_{cm} \right) + \sum_{i=1}^{N} (\mathbf{r}_{cm,i} \times m_i \mathbf{v}_{cm,i}) + \left( \sum_{i=1}^{N} m_i \mathbf{r}_{cm,i} \right) \times \mathbf{v}_{cm} \tag{1.4.4}
\]

The first and third terms in Eq. (1.4.4) are both zero due to the fact that

\[
\sum_{i=1}^{N} m_i \mathbf{r}_{cm,i} = 0 \tag{1.4.5}
\]
\[
\sum_{i=1}^{N} m_i \mathbf{v}_{cm,i} = 0
\]

We first show that \( \sum_{i=1}^{N} m_i \mathbf{r}_{cm,i} \) is zero. We begin by using Eq. (1.2.3)

\[
\sum_{i=1}^{N} (m_i \mathbf{r}_{cm,i}) = \sum_{i=1}^{N} (m_i (\mathbf{r}_i - \mathbf{R}_{cm})) = \sum_{i=1}^{N} m_i \mathbf{r}_i - \left( \sum_{i=1}^{N} m_i \right) \mathbf{R}_{cm} = \sum_{i=1}^{N} m_i \mathbf{r}_i - m_{\text{tot}} \mathbf{R}_{cm} \tag{1.4.6}
\]

Substitute the definition of the center of mass (Eq. (1.2.1)) into Eq. (1.4.6) yielding

\[
\sum_{i=1}^{N} (m_i \mathbf{r}_{cm,i}) = \sum_{i=1}^{N} m_i \mathbf{r}_i - m_{\text{tot}} \frac{1}{m_{\text{tot}}} \sum_{i=1}^{N} m_i \mathbf{r}_i = 0 \tag{1.4.7}
\]

The vanishing of \( \sum_{i=1}^{N} m_i \mathbf{v}_{cm,i} = 0 \) follows directly from the definition of the center of mass frame, that the total momentum in the center of mass is zero. Equivalently the derivative of Eq. (1.4.7) is zero. We could also simply calculate and find that
\[
\sum_{i} m_{i} \vec{v}_{cm,i} = \sum_{i} m_{i} (\vec{v}_{i} - \vec{v}_{cm}) \\
= \sum_{i} m_{i} \vec{v}_{i} - \vec{v}_{cm} \sum_{i} m_{i} \\
= m_{\text{total}} \vec{v}_{cm} - \vec{v}_{cm} m_{\text{total}} \\
= \vec{0}.
\]

We can now simplify Eq. (1.4.4) for the angular momentum about the point \( S \) using the fact that \( \vec{R}_{cm} = \vec{r}_{S,cm} \) (this way we no longer need to specify that \( S \) is at the origin of the reference frame \( O \) and the above calculation still holds), \( m_{\text{total}} = \sum_{i=1}^{N} m_{i} \), and

\[ \vec{p}_{\text{sys}} = m_{\text{total}} \vec{V}_{cm} \](in reference frame \( O \)):

\[
\vec{L}_{S}^{\text{total}} = \vec{r}_{S,cm} \times \vec{p}_{\text{sys}} + \sum_{i=1}^{N} (\vec{r}_{cm,i} \times m_{i} \vec{v}_{cm,i}). \tag{1.4.9}
\]

Consider the first term in Equation (1.4.9), \( \vec{r}_{S,cm} \times \vec{p}_{\text{sys}} \); the vector \( \vec{r}_{S,cm} \) is the vector from the point \( S \) to the center of mass. If we treat the system as a point-like particle of mass \( m_{\text{total}} \) located at the center of mass, then the momentum of this point-like body is \( \vec{p}_{\text{sys}} = m_{\text{total}} \vec{V}_{cm} \). Thus the first term is the angular momentum about the point \( S \) of this point-like body, which is called the \textit{orbital angular momentum} about \( S \),

\[ \vec{L}_{S}^{\text{orbital}} = \vec{r}_{S,cm} \times \vec{p}_{\text{sys}}. \tag{1.4.10} \]

for this system of particles.

Consider the second term in Equation (1.4.9), \( \sum_{i=1}^{N} (\vec{r}_{cm,i} \times m_{i} \vec{v}_{cm,i}) \); the quantity inside the summation is the angular momentum of particle \( i \) with respect to the origin in the center of mass reference frame \( O_{cm} \) (recall the origin in the center of mass reference frame is the center of mass of the system).

\[ \vec{L}_{cm,i} = \vec{r}_{cm,i} \times m_{i} \vec{v}_{cm,i}. \tag{1.4.11} \]

Hence the total angular momentum of the system with respect to the center of mass in the center of mass reference frame is given by
\[ \vec{L}_{\text{cm}}^{\text{spin}} = \sum_{i=1}^{N} \vec{L}_{\text{cm},i} = \sum_{i=1}^{N} \left( \vec{r}_{\text{cm},i} \times m_i \vec{v}_{\text{cm},i} \right). \]  

(1.4.12)

a vector quantity we call the spin angular momentum. Thus we see that the total angular momentum about the point \( S \) is the sum of these two terms,

\[ \vec{L}_{S}^{\text{total}} = \vec{L}_{S}^{\text{orbital}} + \vec{L}_{\text{cm}}^{\text{spin}}. \]  

(1.4.13)

This decomposition of angular momentum into a piece associated with the translational motion of the center of mass and a second piece associated with the rotational motion about the center of mass in the center of mass reference frame is the key conceptual foundation for what follows.

### 1.6 Translational and Rotational Equations of Motion

Recall that it is always true that

\[ \vec{r}_{S}^{\text{total}} = \frac{d\vec{r}_{S}^{\text{total}}}{dt}. \]  

(1.5.1)

We differentiate the LHS of Eq. (1.4.9) and find that

\[ \vec{r}_{S}^{\text{total}} = \frac{d}{dt} \left( \vec{r}_{S,\text{cm}} \times \vec{p}_{\text{sys}} \right) + \frac{d}{dt} \left( \sum_{i=1}^{N} \left( \vec{r}_{\text{cm},i} \times m_i \vec{v}_{\text{cm},i} \right) \right). \]  

(1.5.2)

We apply the vector identity to Eq. (1.5.2)

\[ \frac{d}{dt} \left( \vec{A} \times \vec{B} \right) = \frac{d\vec{A}}{dt} \times \vec{B} + \vec{A} \times \frac{d\vec{B}}{dt}, \]  

(1.5.3)

to Eq. (1.5.2) yielding

\[ \vec{r}_{S}^{\text{total}} = \frac{d\vec{r}_{S,\text{cm}}}{dt} \times \vec{p}_{\text{sys}} + \frac{d\vec{p}_{\text{sys}}}{dt} \times \vec{r}_{S,\text{cm}} \]

\[ + \sum_{i=1}^{N} \left( \frac{d\vec{r}_{\text{cm},i}}{dt} \times m_i \vec{v}_{\text{cm},i} \right) + \sum_{i=1}^{N} \left( \vec{r}_{\text{cm},i} \times \frac{d}{dt} (m_i \vec{v}_{\text{cm},i}) \right). \]  

(1.5.4)

The first and third terms in Equation (1.5.4) are eliminated by noting that
Therefore the time derivative of the angular momentum about a point \( S \), Equation (1.5.4), becomes

\[
\vec{\tau}_S^{\text{total}} = \vec{r}_{S, cm} \times \vec{p}^{\text{sys}} + \sum_{i=1}^{N} \left( \vec{r}_{cm, i} \times m_i \vec{v}_{cm, i} \right).
\] (1.5.6)

In the first term in Equation (1.5.6), the time derivative of the total momentum of the system is the total external force,

\[
\vec{F}^{\text{total}}_{\text{ext}} = \frac{d\vec{p}^{\text{sys}}}{dt}
\] (1.5.7)

and in the second term, the force acting on the element is

\[
\vec{F}_{i} = \frac{d}{dt} (m_i \vec{v}_{cm, i})
\] (1.5.8)

The expression in Equation (1.5.6) then becomes

\[
\vec{\tau}_S^{\text{total}} = \vec{r}_{S, cm} \times \vec{F}^{\text{total}}_{\text{ext}} + \sum_{i=1}^{N} \left( \vec{r}_{cm, i} \times \vec{F}_{i} \right).
\] (1.5.9)

The first term is the contribution to the torque about the point \( S \) if all the external forces were to act at the center of mass. The second term, \( \sum_{i=1}^{N} (\vec{r}_{cm, i} \times \vec{F}_{i}) \), is the sum of the torques on the individual particles in the center of mass reference frame. Then Equation (1.5.9) may be expressed as

\[
\vec{\tau}_S^{\text{total}} = \vec{r}_{S, cm} \times \vec{F}^{\text{total}}_{\text{ext}} + \vec{\tau}_{cm}^{\text{total}}.
\] (1.5.10)

Retracing our calculating we have two conditions. For a system of particles, when we calculate the torque about a point \( S \), we treat the system as a point-like particle located at the center of mass of the system. All the external forces \( \vec{F}^{\text{total}}_{\text{ext}} \) act at the center of mass. We calculate the orbital angular momentum of the center of mass and determine it’s time derivative and then apply
\begin{equation}
\vec{r}_{s, \text{cm}} \times \vec{F}_{\text{ext}} = \frac{dL_{s}^{\text{orbital}}}{dt}. \tag{1.5.11}
\end{equation}

In addition, we calculate the torque about the center of mass due to all the forces acting on the particles in the center of mass reference frame. We calculate the torque is equal to the time derivative of the total angular momentum of the system with respect to the center of mass in the center of mass reference frame and then apply

\begin{equation}
\vec{\tau}_{\text{cm}}^{\text{total}} = \frac{dL_{\text{cm}}^{\text{spin}}}{dt}. \tag{1.5.12}
\end{equation}

### 1.7 Kinetic Energy of a System of Particles

Consider a system of particles. The \textit{i}th particle has mass \(m_i\) and velocity \(\vec{v}_i\) with respect to a reference frame \(O\). The kinetic energy of the system of particles is given by

\begin{equation}
K = \frac{1}{2} \sum_i m_i \vec{v}_i^2 = \frac{1}{2} \sum_i m_i \vec{v}_i \cdot \vec{v}_i
= \frac{1}{2} \sum_i m_i (\vec{\nu}_{\text{cm},i} + \vec{\nu}_{\text{cm}}) \cdot (\vec{\nu}_{\text{cm},i} + \vec{\nu}_{\text{cm}}) \tag{1.6.1}
\end{equation}

where Equation (1.2.7) has been used to express \(\vec{v}_i\) in terms of \(\vec{\nu}_{\text{cm},i}\) and \(\vec{\nu}_{\text{cm}}\).

Expanding the last dot product in Equation (1.6.1),

\begin{align*}
K &= \frac{1}{2} \sum_i m_i (\vec{\nu}_{\text{cm},i} \cdot \vec{\nu}_{i, \text{rel}} + \vec{\nu}_{\text{cm}} \cdot \vec{\nu}_{\text{cm}} + 2 \vec{\nu}_{\text{cm},i} \cdot \vec{\nu}_{\text{cm}}) \\
&= \frac{1}{2} \sum_i m_i (\vec{\nu}_{\text{cm},i} \cdot \vec{\nu}_{i, \text{rel}}) + \frac{1}{2} \sum_i m_i (\vec{\nu}_{\text{cm}} \cdot \vec{\nu}_{\text{cm}}) + \sum_i m_i \vec{\nu}_{\text{cm},i} \cdot \vec{\nu}_{\text{cm}} \tag{1.6.2}
\end{align*}

\begin{align*}
&= \sum_i \frac{1}{2} m_i \nu_{\text{cm},i}^2 + \frac{1}{2} \sum_i m_i \nu_{\text{cm}}^2 + \left( \sum_i m \vec{\nu}_{\text{cm},i} \right) \cdot \vec{\nu}_{\text{cm}}
\end{align*}

The last term in the third equation in (1.6.2) vanishes as we showed in Eq. (1.4.5). Then Equation (1.6.2) reduces to

\begin{equation}
K = \sum_i \frac{1}{2} m_i \nu_{\text{cm},i}^2 + \frac{1}{2} \sum_i m_i \nu_{\text{cm}}^2 \tag{1.6.3}
= \sum_i \frac{1}{2} m_i \nu_{\text{cm},i}^2 + \frac{1}{2} m_{\text{total}} \nu_{\text{cm}}^2.
\end{equation}
We interpret the first term as the kinetic energy of the center of mass motion in reference frame $O$ and the second term as the sum of the individual kinetic energies of the particles of the system in the center of mass reference frame $O_{cm}$.

At this point, it’s important to note that no assumption was made regarding the mass elements being constituents of a rigid body. Equation (1.6.3) is valid for a rigid body, a gas, a firecracker (but $K$ is certainly not the same before and after detonation), the sixteen pool balls after the break, or any collection of objects for which the center of mass can be determined.

1.8 Translation and Rotation of a Rigid Body Undergoing Fixed Axis Rotation

For the special case of rigid body of mass $m$, we showed that with respect to a reference frame in which the center of mass of the rigid body is moving with velocity $\mathbf{V}_{cm}$, all elements of the rigid body are rotating about the center of mass with the same angular velocity $\omega_{cm}$.

For the rigid body of mass $m$ and momentum $\mathbf{p} = m\mathbf{V}_{cm}$, the translational equation of motion is still given by Eq. (1.3.1) which we repeat in the form

$$\mathbf{F}_{ext}^{\text{total}} = m\mathbf{A}_{cm}.$$  \hspace{1cm} (1.7.1)

Let’s choose the $z$-axis as the axis of rotation that passes through the center of mass of the rigid body. We have already seen in our discussion of angular momentum of a rigid body that the angular momentum does not necessarily point in the same direction as the angular velocity. However we can take the $z$-component of Eq. (1.5.12)

$$\left(\mathbf{L}_{cm}^{\text{spin}}\right)_z = \frac{d(\mathbf{L}_{cm}^{\text{spin}})_z}{dt}.$$ \hspace{1cm} (1.7.2)

For a rigid body rotating about the center of mass with $\tilde{\omega}_{cm} = \omega_{cm,z} \hat{k}$, the $z$-component of angular momentum about the center of mass is

$$\left(\mathbf{L}_{cm}^{\text{spin}}\right)_z = I_{cm} \omega_{cm,z} \hat{k}.$$ \hspace{1cm} (1.7.3)

The our rotational equation of motion is

$$\left(\mathbf{\tau}_{cm}^{\text{total}}\right)_z = I_{cm} \frac{d\omega_{cm,z}}{dt} = I_{cm} \alpha_{cm,z}.$$ \hspace{1cm} (1.7.4)

1.7.1 Example: Earth’s motion around the sun
The earth, of mass \( m_e = 5.97 \times 10^{24} \text{ kg} \) and (mean) radius \( R_e = 6.38 \times 10^6 \text{ m} \), moves in a nearly circular orbit of radius \( r_{e, \text{orb}} = 1.50 \times 10^{11} \text{ m} \) around the sun with a period \( T_{\text{orb}} = 365.25 \text{ days} \), and spins about its axis in a period \( T_{\text{spin}} = 23 \text{ hr 56 min} \), the axis inclined to the normal to the plane of its orbit around the sun by 23.5° (Figure 15.13).

**Figure 15.13** Orbital and spin motion of the earth.

If we approximate the earth as a uniform sphere, then the moment of inertia of the earth about its center of mass is

\[
I_{\text{cm}} = \frac{2}{5} m_e R_e^2
\]

(a derivation of Equation (1.7.5) is given in the Appendix to this chapter). If we choose the point \( S \) to be at the center of the sun, and assume the orbit is circular, then the orbital angular momentum is

\[
\mathbf{L}_{\text{orbital}} = \mathbf{r}_{S, \text{cm}} \times \mathbf{p} = r_{e, \text{orb}} \mathbf{v}_{\text{orbital}} = r_{e, \text{orb}} m_e \mathbf{v}_{\text{cm}} \mathbf{\hat{k}}.
\]

The velocity of the center of mass of the earth about the sun is related to the orbital angular velocity by

\[
\mathbf{v}_{\text{cm}} = r_{e, \text{orb}} \Omega_{\text{orbit}},
\]

where the orbital angular velocity is

\[
\Omega_{\text{orbit}} = \frac{2\pi}{T_{\text{orb}}} = \frac{2\pi}{(365.25 \text{ d})(8.640 \times 10^4 \text{ s \cdot d}^{-1})} = 1.991 \times 10^{-7} \text{ rad \cdot s}^{-1}.
\]

The orbital angular momentum is then
The spin angular momentum is given by

\[
\mathbf{L}_{\text{spin}} = I_{\text{cm}} \mathbf{\omega}_{\text{spin}} = \frac{2}{5} m_e R_e^2 \mathbf{\omega}_{\text{spin}} \mathbf{\hat{n}},
\]  

(1.7.10)

where \( \mathbf{\hat{n}} \) is a unit normal pointing along the axis of rotation of the earth and

\[
\omega_{\text{spin}} = \frac{2\pi}{T_{\text{spin}}} = \frac{2\pi}{8.616 \times 10^4 \text{ s}} = 7.293 \times 10^{-5} \text{ rad \cdot s}^{-1}.
\]  

(1.7.11)

The spin angular momentum is then

\[
\mathbf{L}_{\text{spin}} = \frac{2}{5} (5.97 \times 10^{24} \text{ kg})(6.38 \times 10^6 \text{ m})^2 (7.293 \times 10^{-5} \text{ rad \cdot s}^{-1}) \mathbf{\hat{n}}
\]  

(1.7.12)

\[
= (7.10 \times 10^{33} \text{ kg \cdot m}^2 \cdot \text{s}^{-1}) \mathbf{\hat{n}}.
\]

The ratio of the magnitudes of the orbital angular momentum to the spin angular momentum is greater than a million,

\[
\frac{L_{\text{orbital}}}{L_{\text{spin}}} = \frac{m_e r_{s,e}^2 \omega_{\text{orbit}}}{(2/5) m_e R_e^2 \omega_{\text{spin}}} = \frac{5 r_{s,e}^2 T_{\text{spin}}}{2 R_e^2 T_{\text{orbit}}} = 3.77 \times 10^6,
\]  

(1.7.13)

as this ratio is proportional to the square of the ratio of the distance to the sun to the radius of the earth.

The total angular momentum is then

\[
\mathbf{L}_{\text{total}} = m_e r_{s,e}^2 \omega_{\text{orbit}} \mathbf{\hat{k}} + \frac{2}{5} m_e R_e^2 \mathbf{\omega}_{\text{spin}} \mathbf{\hat{n}}.
\]  

(1.7.14)

Two finer points should be noted in the above calculations. First, all numerical calculations were performed by computer, keeping far more significant figures than needed in the intermediate calculations. The orbit and spin periods are known to far more precision than the average values used for the earth’s orbit radius and mean radius. Second, two different values have been used for one “day;” in converting the orbit period from days to seconds, the value for the solar day, \( T_{\text{Solar}} = 86,400 \text{ s} \) was used. In converting the earth’s spin angular frequency, the sidereal day, \( T_{\text{Sidereal}} = T_{\text{Spin}} = 86,160 \text{ s} \) was used. The two periods, the solar day from noon to noon and the sidereal day from
the difference between the times that a fixed star is at the same place in the sky, do differ in the third significant figure. See Astrophysical Constants and Parameters.

### 1.9 Rotational Kinetic Energy for a Rigid Body Undergoing Fixed Axis Rotation

The rotational kinetic energy for the rigid body, using \( \vec{v}_{\text{cm,i}} = (r_{\text{cm,i}}) \cdot \vec{\omega}_{\text{cm}} \cdot \vec{\theta} \), simplifies to

\[
K_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2. 
\] (1.8.1)

Therefore the total kinetic energy of a translating and rotating rigid body is

\[
K_{\text{total}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2} m V_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2. 
\] (1.8.2)

### 1.10 Work-Energy Theorem

For a rigid body, we can also consider the work-energy theorem separately for the translational motion and the rotational motion. Once again treat the rigid body as a point-like particle moving with velocity \( \vec{V}_{\text{cm}} \) in reference frame \( O \). We can use the same technique that when treating point particles to show that the work done by the external forces is equal to the change in kinetic energy

\[
W_{\text{ext}} = \int_{0}^{f} \vec{F}_{\text{ext}} \cdot d\vec{r} = \int_{0}^{f} \frac{d(m \vec{V}_{\text{cm}})}{dt} \cdot d\vec{R}_{\text{cm}} = m \int_{0}^{f} \frac{d(\vec{V}_{\text{cm}})}{dt} \cdot \vec{V}_{\text{cm}} dt = \int_{0}^{f} m \vec{V}_{\text{cm}} \cdot d\vec{V}_{\text{cm}} = \int_{0}^{f} d\vec{V}_{\text{cm}} = \Delta K_{\text{trans}}.
\] (1.9.1)

Thus for the translational motion

\[
\int_{0}^{f} \vec{F}_{\text{ext}} \cdot d\vec{r} = \frac{1}{2} m V_{\text{cm,f}}^2 - \frac{1}{2} m V_{\text{cm,i}}^2. 
\] (1.9.2)

For the rotational motion we go the center of mass reference frame and the rotational work done i.e. the integral of the z-component of the torque about the center of mass with respect to \( d\theta \). Then

\[
\int_{0}^{f} (\vec{T}_{\text{cm}})_{z} d\theta = \int_{0}^{f} I_{\text{cm}} \omega_{\text{cm}} d\theta = \int_{0}^{f} I_{\text{cm}} \omega_{\text{cm}} d\theta = \int_{0}^{f} I_{\text{cm}} d\omega_{\text{cm}} \omega_{\text{cm}} = \Delta K_{\text{rot}}.
\] (1.9.3)
Therefore we can combine these two separate results Eqs. (1.9.2) and (1.9.3) and get the work-energy theorem for a rotating and translating rigid body that undergoes fixed axis rotation about the center of mass.

\[
W_{o,f}^{\text{total}} = \int_0^f \mathbf{F}_{\text{ext}} \cdot d\mathbf{r} + \int_0^f (\mathbf{r}_{cm}^{\text{total}})' d\theta
\]

\[
= \left( \frac{1}{2} m V_{cm,f}^2 + \frac{1}{2} I_{cm} \omega_{cm,f}^2 \right) - \left( \frac{1}{2} m V_{cm,i}^2 + \frac{1}{2} I_{cm} \omega_{cm,i}^2 \right)
\]

\[
= \Delta K_{\text{trans}} + \Delta K_{\text{rot}} = \Delta K_{\text{total}}
\] (1.9.4)

So Eqs. (1.7.1), (1.7.4), and (1.9.4) are principles that we shall employ to analyze the motion of a rigid bodies undergoing translation and fixed axis rotation about the center of mass. (We shall call these our “Rules to Live By”.)

### 1.11 Worked Examples

#### 1.10.1 Example Angular Impulse

Two point-like objects are located at the points A, and B, of respective masses \( M_A = 2M \), and \( M_B = M \), as shown in the figure below. The two objects are initially oriented along the y-axis and connected by a rod of negligible mass of length \( D \), forming a rigid body. A force of magnitude \( F = \mathbf{F} \) along the x direction is applied to the object at B at \( t = 0 \) for a short time interval \( \Delta t \). Neglect gravity. Give all your answers in terms of \( M \) and \( D \) as needed.

![Diagram of two point-like objects connected by a rod](image)

a. Describe qualitatively in words how the system moves after the force is applied: direction, translation and rotation.

b. How far is the center of mass of the system from the object at point B?
c. What is the direction and magnitude of the linear velocity of the center-of-mass after the collision?

d. What is the magnitude of the angular velocity of the system after the collision?

e. Is it possible to apply another force of magnitude \( F \) along the positive \( x \) direction to prevent the system from rotating? Does it matter where the force is applied?

f. Is it possible to apply another force of magnitude \( F \) in some direction to prevent the center of mass from translating? Does it matter where the force is applied?

Solutions:

a) An impulse of magnitude \( F \Delta t \) is applied in the \( +x \) direction, and the center of mass of the system will move in this direction. The two masses will rotate about the center of mass, counterclockwise in the figure.

b) Before the force is applied we can calculate the position of the center of mass

\[
\mathbf{\vec{R}_{cm}} = \frac{M_A \mathbf{\vec{r}_A} + M_B \mathbf{\vec{r}_B}}{M_A + M_B} = \frac{2M(D/2) \mathbf{\hat{j}} + M(D/2)(-\mathbf{\hat{j}})}{3M} = \frac{D}{6} \mathbf{\hat{j}}.
\]

The center of mass is a distance \((2/3)D\) from the object at B and is a distance \((1/3)D\) from the object at A.

c) Because \( F\Delta t = 3M\dot{V}_{cm} \), the magnitude of the velocity of the center of mass is then \((F\Delta t)/(3M)\) and the direction is in the positive \( \mathbf{\hat{i}} \) direction (to the right).
d) Because the force is applied at the point B, there is no torque about the point B, hence the angular momentum is constant about the point B. The initial angular momentum about the point B is zero.

The angular momentum about the point B after the impulse is applied is the sum of two terms,

$$\mathbf{\hat{\omega}} = \mathbf{\hat{L}}_{B, f} = \mathbf{\hat{r}}_{B, f} \times 3M \mathbf{\hat{V}}_{cm} + \mathbf{\hat{L}}_{cm} = (2D/3) \mathbf{\hat{j}} \times F \Delta t \mathbf{\hat{i}} + \mathbf{\hat{L}}_{cm}$$

$$\mathbf{\hat{L}}_{cm} = I_{cm} \omega \mathbf{\hat{k}} = (2M(D/3)^2 + M(2D/3)^2) \omega \mathbf{\hat{k}} = (2/3)MD^2 \omega \mathbf{\hat{k}}$$

Thus the angular about the point B after the impulse is applied is

$$\mathbf{\hat{\omega}} = (2DF\Delta t/3)(-\mathbf{\hat{k}}) + (2/3)MD^2 \omega \mathbf{\hat{k}}.$$

We can solve this equation for the angular speed

$$\omega = \frac{F\Delta t}{MD}$$

e) No. The force additional force would have to be applied at a distance $2D/3$ above the center of mass, which is not a physical point of the system.

f) An additional force of the same magnitude, in the negative x direction, would result in no net force and hence no acceleration of the center of mass.
1.10.2 Example Person on a railroad car moving in a circle

A person of mass $M$ is standing on a railroad car, which is rounding an unbanked turn of radius $R$ at a speed $v$. His center of mass is at a height of $L$ above the car midway between his feet, which are separated by a distance of $d$. The man is facing the direction of motion. What is the magnitude of the normal force on each foot?

![Diagram of a person on a railroad car](image)

**Solution:** We begin by choosing a cylindrical coordinate system and drawing a free-body force diagram, shown below.

![Free-body force diagram](image)

We decompose the contact force between the foot closest to the center of the circular motion and the ground into a tangential component corresponding to static friction $\mathbf{f}_1$ and a perpendicular component, $\mathbf{N}_1$. In a similar fashion we decompose the contact force between the foot furthest from the center of the circular motion and the ground into a tangential component corresponding to static friction $\mathbf{f}_2$ and a perpendicular component, $\mathbf{N}_2$. We do not assume that the static friction has its maximum magnitude nor do we assume that $\mathbf{f}_1 = \mathbf{f}_2$ or $\mathbf{N}_1 = \mathbf{N}_2$. The gravitational force acts at the center of mass.

We shall use our two dynamical equations of motion, Eq. (1.7.1) for translational motion and Eq. (1.7.4) for the rotational motion about the center of mass noting that we are considering the special case that $\mathbf{a}_{\text{cm}} = 0$ because the object is not rotating about the center of mass.
In order to apply Eq. (1.7.1), we treat the person as a point-like particle located at the center of mass and all the eternal forces act at this point. The radial component of Newton’s Second Law (Eq. (1.7.1)) is given by

\[ \hat{r} : -f_1 - f_2 = -m \frac{v^2}{R}. \] (1.10.1)

The vertical component of Newton’s Second Law is given by

\[ \hat{k} : N_1 + N_2 - mg = 0. \] (1.10.2)

The rotational equation of motion (Eq. (1.7.4)) is

\[ \tau_{\text{total}}^{\text{cm}} = 0. \] (1.10.3)

We begin our calculation of the torques about the center of mass by noting that the gravitational force does not contribute to the torque because it is acting at the center of mass. We draw a torque diagram in the figure below showing the location of the point of application of the forces, the point we are computing the torque about (which in this case is the center of mass), and the vector \( \hat{r}_{\text{cm},l} \) from the point we are computing the torque about to the point of application of the forces.

The torque on the inner foot is given by

\[ \tau_{\text{cm},l} = \hat{r}_{\text{cm},l} \times (\hat{f}_1 + \hat{N}_1) = \left( -\frac{d}{2} \hat{r} - L \hat{k} \right) \times (-f_1 \hat{r} + N_1 \hat{k}) = \left( \frac{d}{2} N_1 + Lf_1 \right) \hat{\theta}. \] (1.10.4)

We draw a similar torque diagram for the forces applied to the outer foot.
The torque on the outer foot is given by

\[ \vec{\tau}_{cm,2} = \vec{r}_{cm,2} \times (\vec{f}_2 + \vec{N}_2) = \left( \frac{d}{2} \hat{r} - L\hat{k} \right) \times (\vec{f}_2 + \vec{N}_2) = \left( -\frac{d}{2} N_2 + Lf_2 \right) \hat{\theta}. \]

Notice that the forces \( \vec{f}_1, \vec{N}_1, \) and \( \vec{f}_2 \) all contribute torques about the center of mass in the positive \( \hat{\theta} \)-direction while \( \vec{N}_2 \) contribute torques about the center of mass in the negative \( \hat{\theta} \)-direction while \( \vec{N}_2 \). According to Eq. (1.10.3) the sum of these torques about the center of mass must be zero. Therefore

\[ \vec{\tau}^{\text{total}}_{cm} = \vec{\tau}_{cm,1} + \vec{\tau}_{cm,2} = \left( \frac{d}{2} N_1 + Lf_1 \right) \hat{\theta} + \left( -\frac{d}{2} N_2 + Lf_2 \right) \hat{\theta} = \left( \frac{d}{2} (N_1 - N_2) + L(f_1 + f_2) \right) \hat{\theta} = 0. \]

Notice that the magnitudes of the two friction forces appear together as a sum in Eqs. (1.10.6) and (1.10.1). We now can solve Eq. (1.10.1) for \( f_1 + f_2 \) and substitute the result into Eq. (1.10.6) yielding the condition that

\[ \frac{d}{2} (N_1 - N_2) + Lm \frac{v^2}{R} = 0. \]

We can rewrite this equation as

\[ N_2 - N_1 = \frac{2Lmv^2}{dR}. \]

We also rewrite the vertical equation of motion (Eq. (1.10.2) in the form

\[ N_2 + N_1 = mg. \]

We now can solve for \( N_2 \) by adding together Eqs. (1.10.8) and (1.10.9) and then divide by two,
\[ N_2 = \frac{1}{2} \left( \frac{2Lmv^2}{dR} + mg \right). \]  

(1.10.10)

We now can solve for \( N_1 \) by subtracting Eqs. (1.10.8) from (1.10.9) and then divide by two,

\[ N_1 = \frac{1}{2} \left( mg - \frac{2Lmv^2}{dR} \right). \]  

(1.10.11)

Check your result: We see that the normal force acting on the outer foot is greater in magnitude than the normal force acting on the inner foot. We expect this result because as we increase the speed \( v \), we find that at a maximum speed \( v_{\text{max}} \), the normal force on the inner foot goes to zero and we start to rotate in the positive \( \hat{\theta} \)-direction, tipping outward. We can find this maximum speed by setting \( N_1 = 0 \) in Eq. (1.10.11) resulting in

\[ v_{\text{max}} = \sqrt{\frac{gdR}{2L}}. \]  

(1.10.12)

### 1.10.3 Example Torque, Rotation and Translation: Yo-Yo

A Yo-Yo of mass \( m \) has an axle of radius \( b \) and a spool of radius \( R \). Its moment of inertia about the center can be taken to be \( I_{cm} = \frac{1}{2}mR^2 \) and the thickness of the string can be neglected. The Yo-Yo is released from rest. You will need to assume that the center of mass of the Yo-Yo descends vertically, and that the string is vertical as it unwinds.

1. **What is the tension in the cord as the Yo-Yo descends?**
2. **What is the magnitude of the angular acceleration as the yo-yo descends and the magnitude of the linear acceleration?**
3. **Find the angular velocity of the Yo-Yo when it reaches the bottom of the string, when a length \( l \) of the string has unwound.**
Solutions:

a) As the Yo-Yo descends it rotates clockwise in the above diagram. The torque about the center of mass of the Yo-Yo is due to the tension and increases the magnitude of the angular velocity (see the figure below).

The direction of the torque is into the page in the figure above (positive $z$-direction). Use the right-hand rule to check this, or use the cross-product definition of torque:

$$\vec{\tau}_{\text{cm}} = \vec{r}_{\text{cm},r} \times \vec{T}. \quad (1.10.13)$$

About the center of mass, $\vec{r}_{\text{cm},r} = -b \hat{i}$ and $\vec{T} = -T \hat{j}$, so the torque is

$$\vec{\tau}_{\text{cm}} = \vec{r}_{\text{cm},r} \times \vec{T} = (-b \hat{i}) \times (-T \hat{j}) = bT \hat{k}. \quad (1.10.14)$$

Applying Newton’s Second Law in the $\hat{j}$-direction,

$$mg - T = ma_y \quad (1.10.15)$$

Applying the torque equation for the Yo-Yo:

$$bT = l_{\text{cm}} \alpha_z \quad (1.10.16)$$

where $\alpha$ is the $z$-component of the angular acceleration.

The $z$-component of the angular acceleration and the $y$-component of the linear acceleration are related by the constraint condition

$$a_y = b \alpha_z \quad (1.10.17)$$

where $b$ is the axle radius of the Yo-Yo. Substitute Eq. $(1.10.17)$ into $(1.10.15)$ yielding
\[ mg - T = mb\alpha_z \]  \hspace{1cm} (1.10.18)

Now solve Eq. (1.10.16) for \( \alpha_z \) and substitute the result into Eq. (1.10.18),

\[ mg - T = \frac{mb^2 T}{I_{cm}} \]  \hspace{1cm} (1.10.19)

Solve Eq. (1.10.19) for the tension \( T \),

\[ T = \frac{mg}{1 + \frac{mb^2}{I_{cm}}} = \frac{mg}{1 + \frac{mb^2}{(1/2)mR^2}} = \frac{mg}{1 + \frac{2b^2}{R^2}} \]  \hspace{1cm} (1.10.20)

b) Substitute Eq. (1.10.20) into Eq. (1.10.16) to determine the \( z \)-component of the angular acceleration,

\[ \alpha_z = \frac{bT}{I_{cm}} = \frac{2bg}{(R^2 + 2b^2)} \]  \hspace{1cm} (1.10.21)

Using the constraint condition Eq. (1.10.17), we can find the \( y \)-component of linear acceleration is then

\[ a_y = b\alpha_z = \frac{2b^2 g}{(R^2 + 2b^2)} = \frac{g}{1 + R^2 / 2b^2} \]  \hspace{1cm} (1.10.22)

Note that both quantities \( \alpha_z > 0 \) and so Eqs. (1.10.21) and (1.10.22) are the magnitudes of the respective quantities. For a typical Yo-Yo, the acceleration is much less than that of an object in free fall.

b) Use conservation of energy to determine the angular velocity of the Yo-Yo when it reaches the bottom of the string. As in the above diagram, choose the downward vertical direction as the positive \( \hat{j} \)-direction and let \( y = 0 \) designate the location of the center of mass of the Yo-Yo when the string is completely wound. Choose \( U(y = 0) = 0 \) for the zero reference potential energy. This choice of direction and reference means that the gravitational potential energy will be negative but increasing while the Yo-Yo descends. For this case, the gravitational potential energy is

\[ U = -mg \cdot y \]  \hspace{1cm} (1.10.23)
Mechanical energy in the initial state (Yo-Yo is completely wound): the Yo-Yo is not yet moving downward or rotating, and the center of mass is located at $y = 0$ so the mechanical energy is zero

$$E_i = 0.$$  \hfill (1.10.24)

Mechanical energy in the final state (Yo-Yo is completely unwound): Denote the linear speed of the Yo-Yo as $v_f$ and its angular speed as $\omega_f$ (at the point $y = l$). The constraint condition between $v_f$ and $\omega_f$ is given by

$$v_f = b\omega_f,$$  \hfill (1.10.25)

consistent with Eq. (1.10.17). The kinetic energy is the sum of translational and rotational kinetic energy, where we have used $I_{cm} = (1/2)mr^2$,

$$E_f = K_f + U_f = \frac{1}{2}mv_f^2 + \frac{1}{2}I_{cm}\omega_f^2 - mgl$$

$$= \frac{1}{2}mb^2\omega_f^2 + \frac{1}{4}mR^2\omega_f^2 - mgl.$$  \hfill (1.10.26)

There are no external forces doing work on the system, so

$$0 = E_f = E_i$$  \hfill (1.10.27)

Thus

$$\left(\frac{1}{2}mb^2 + \frac{1}{4}mR^2\right)\omega_f^2 = mgl.$$  \hfill (1.10.28)

Solving for $\omega_f$,

$$\omega_f = \sqrt{\frac{4gl}{(2b^2 + R^2)}}.$$  \hfill (1.10.29)

Note: We could also use kinematics to determine the final angular velocity by solving for the time interval $\Delta t$ that it takes for the Yo-Yo to travel a distance $l$ at the constant acceleration found in Eq. (1.10.22)),

$$\Delta t = \sqrt{\frac{2l}{a_y}} = \sqrt{\frac{l(R^2 + 2b^2)}{b^2g}}.$$  \hfill (1.10.30)
The final angular velocity of the Yo-Yo is then (using Eq. (1.10.21) for the z-component of the angular acceleration),

$$\omega_f = \alpha_z \Delta t = \sqrt[1.10.31]{\frac{4gl}{(R^2 + 2b^2)}},$$

in agreement with Eq. (1.10.29).

1.10.4 Example Bowling Ball

A bowling ball of mass $m$ and radius $R$ is initially thrown down an alley with an initial speed $v_0$, and it slides without rolling but due to friction it begins to roll. The moment of inertia of the ball about its center of mass is $I_{cm} = (2/5)mr^2$. What is the speed $v_f$ of the bowling ball when it just start to roll without slipping?

![Diagram of bowling ball with velocities $v_0$ and $v_f$.]

**Solution:** You may want to consider what effect friction has on the bowling ball. In which direction does the frictional force point? While the ball slides is the friction static or kinetic? While the ball rolls without slipping is the friction static or kinetic? We shall solve this problem using two different methods.

1) Relating the torque and angular momentum about the center of mass, applying Newton’s Second Law for the translational motion of the center of mass, and then using kinematics.

2) Applying the condition that the angular momentum is constant about a fixed point lying along the line of contact of the bowling ball and the surface, and applying the rolling without slipping condition.

In both solutions, take positive torque and angular momentum as being *into* the page as shown in the figure below, and take the positive direction of velocity, linear momentum and force to be to the right in the figure.
Torque and angular momentum about the center of mass:

Gravity exerts no torque about the center of mass, and the normal component of the contact force has a zero moment arm; the only force that exerts a torque is the frictional force, with a moment arm of $R$ (the force vector and the radius vector are perpendicular).

The frictional force should be in the negative direction, to the left in the figure above. From the right-hand rule, the direction of the torque is into the page, and hence in the positive $z$-direction. Equating the $z$-component of the torque to the rate of change of angular momentum,

$$\tau_{cm,z} = R f_k = I_{cm} \alpha_z,$$

(1.10.32)

where $f_k$ is the magnitude of the kinetic friction force and $\alpha_z$ is the $z$-component of the angular acceleration of the bowling ball. Note that Equation (1.10.32) results in a positive angular acceleration, which is consistent with the ball tending to rotate as indicated in the figure.

The friction force is also the only force in the horizontal direction, and will cause an acceleration of the center of mass,

$$a_{cm} = -f_{\text{friction}} / m.$$

(1.10.33)

Note that the acceleration will be negative, as expected.

Now we need to consider the kinematics. The bowling ball will increase its angular speed and decrease its linear speed according to
\[
\omega_z(t) = \alpha_z t = \frac{R f_k}{I_{cm}} t \\
v_x(t) = v_{x,0} - \frac{f_k}{m} t.
\] (1.10.34)

As soon as the ball stops slipping, the kinetic friction no longer acts, and the ball moves with constant angular speed and constant linear speed. Denote the time when this happens as \( t_f \); at this time, \( v_{x,f} = R \omega_{z,f} \) and the relations in Equation (1.10.34) become

\[
R^2 \frac{f_k}{I_{cm}} t_f = v_{x,f}.
\] (1.10.35)

\[
v_{x,0} - \frac{f_k}{m} t_f = v_{x,f}.
\]

We can now solve the first equation in Eq. (1.10.35) for \( t_f \) and find that

\[
t_f = \frac{I_{cm}}{f_k R^2} v_f.
\] (1.10.36)

We now substitute Eq. (1.10.36) into the second equation in Eq. (1.10.35) and find that

\[
v_{x,f} = v_{x,0} - \frac{f_k}{m} \frac{I_{cm}}{f_k R^2} v_{x,f}
\] (1.10.37)

\[
v_{x,f} = v_{x,0} - \frac{I_{cm}}{m R^2} v_f.
\]

The second equation in (1.10.37) is easily solved for

\[
v_{x,f} = \frac{v_{x,0}}{1 + \frac{I_{cm}}{m R^2}} = \frac{5}{7} v_{x,0},
\] (1.10.38)

using the given \( I_{cm} = (2/5) m R^2 \) for a uniform sphere.

The above derivation shows that the nature of the frictional force, specifically any coefficient of kinetic friction, does not affect the result. However the time it takes for the ball to roll without slipping will depend on the frictional force.

2) Since the frictional force did not enter into the final result, we should suspect that choosing a different point about which to find torque and angular momentum, one that
would not involve a frictional torque, might make things easier. If we pick a point such that the moment arm of the friction force is zero, the frictional torque is zero.

As the ball moves down the alley, the contact point will move, but the friction force will always be parallel to the line of contact between the bowling bowl and the surface. So, if we pick any fixed point along the line of contact between the bowling bowl and the surface for finding torque and angular momentum, there will be no frictional torque. Gravity and the normal component of the contact force will act with the same moment arm, but in opposite directions, and these two torques will cancel. Hence, there is no net torque and the angular momentum is constant.

The initial angular momentum is only due to the translation of the center of mass,

\[
\mathbf{L}_{P,0} = \mathbf{r}_{P,cm,0} \times m \mathbf{v}_{x,0} \hat{i} = (x_0 \hat{i} - R \hat{j}) \times (m \mathbf{v}_{x,0} \hat{i}) = m R \mathbf{v}_{x,0} \hat{k}, \quad (1.10.39)
\]

where we have chosen \( \hat{k} \) to point into the page. The final angular momentum is both translational and rotational,

\[
\mathbf{L}_{P,f} = \mathbf{r}_{P,cm,f} \times m \mathbf{v}_{x,f} \hat{i} + I_{cm} \omega_f \hat{k} = (x_0 \hat{i} - R \hat{j}) \times (m \mathbf{v}_{x,f} \hat{i})
\]
\[= (m R \mathbf{v}_{x,f} + I_{cm} \mathbf{v}_{x,f} / R) \hat{k}, \quad (1.10.40)
\]

where the condition for rolling without slipping, \( \mathbf{v}_{x,f} = R \omega_{z,f} \) and \( I_{cm} = (2/5)mR^2 \) have been used. Equating the z-components in Eqs. (1.10.39) and (1.10.40) yields

\[
m R \mathbf{v}_{x,0} = (7/5)m R \mathbf{v}_{x,f}. \quad (1.10.41)
\]
which we can solve for final x-component of the velocity of the center of mass of the bowling ball

\[ m R v_{x,0} = \left( \frac{7}{5} \right) m R v_{x,f} . \]  

(1.10.42)

agreeing with our result in Eq. (1.10.38).

### 1.10.5 Example Rotation and Translation: Object and Stick Collision

A long narrow uniform stick of length \( l \) and mass \( m \) lies motionless on ice (assume the ice provides a frictionless surface). The center of mass of the stick is the same as the geometric center (at the midpoint of the stick). The moment of inertia of the stick about its center of mass is \( I_{cm} \). A puck (with putty on one side) has the same mass \( m \) as the stick. The puck slides without spinning on the ice with a speed of \( v_0 \) toward the stick, hits one end of the stick, and attaches to it. You may assume that the radius of the puck is much less than the length of the stick so that the moment of inertia of the puck about its center of mass is negligible compared to \( I_{cm} \).

![Diagram of stick and puck collision](image)

a) How far from the midpoint of the stick is the center of mass of the stick-puck combination after the collision?

b) What is the linear velocity of the stick plus puck after the collision?

b) Is mechanical energy conserved during the collision? Explain your reasoning.

d) What is the angular velocity of the stick plus puck after the collision?

e) How far does the stick's center of mass move during one rotation of the stick?
Solution:

In this problem we will calculate the center of mass of the puck-stick system after the collision. There are no external forces or torques acting on this system so the momentum of the center of mass is constant before and after the collision and the angular momentum about the center of mass of the puck-stick system is constant before and after the collision. We shall use these relations to compute the final angular velocity of the puck-stick about the center of mass. We note that the mechanical energy is not constant because the puck collides completely inelastically with the stick.

a) With respect to the center of the stick, the center of mass of the stick-puck combination is (neglecting the radius of the puck)

\[ d_{\text{cm}} = \frac{m_{\text{stick}} d_{\text{stick}} + m_{\text{puck}} d_{\text{puck}}}{m_{\text{stick}} + m_{\text{puck}}} = \frac{m(l/2)}{m+m} = \frac{l}{4}. \tag{1.10.43} \]

b) During the collision, the only net forces on the system (the stick-puck combination) are the internal forces between the stick and the puck (transmitted through the putty).

Hence, linear momentum is conserved. Initially only the puck had linear momentum \( p_0 = m v_0 \). After the collision, the center of mass of the system is moving with speed \( v_f \). Equating initial and final linear momenta,

\[ m v_0 = (2m) v_f \Rightarrow v_f = \frac{v_0}{2}. \tag{1.10.44} \]

The direction of the velocity is the same as the initial direction of the puck’s velocity.

Note that the result of part a) was not needed for part b); if the masses are the same, Equation (1.10.44) would hold for any mass distribution of the stick.

c) The forces that deform the putty do negative work (the putty is compressed somewhat), and so mechanical energy is not conserved; the collision is totally inelastic.
d) Choose the center of mass of the stick-puck combination, as found in part a), as the point about which to find angular momentum. This choice means that after the collision there is no angular momentum due to the translation of the center of mass. Before the collision, the angular momentum was entirely due to the motion of the puck,

\[ \mathbf{L}_0 = \mathbf{r}_{\text{puck}} \times \mathbf{p}_0 = (l/4)(m v_0)\hat{k}, \]  

(1.10.45)

where \( \hat{k} \) is directed out of the page in the figure above. After the collision, the angular momentum is

\[ \mathbf{L}_f = I_{cm} \omega_f \hat{k}, \]  

(1.10.46)

where \( I_{cm} \) is the moment of inertia about the center of mass of the stick-puck combination. This moment of inertia of the stick about the new center of mass is found from the parallel axis theorem, and the moment of inertia of the puck is \( m(l/4)^2 \), and so

\[ I_{cm} = I_{cm', \text{stick}} + I_{cm', \text{puck}} = (I_{cm} + m(l/4)^2) + m(l/4)^2 = I_{cm} + \frac{ml^2}{8}. \]  

(1.10.47)

Inserting this expression into Equation (1.10.46), equating the expressions for \( \mathbf{L}_0 \) and \( \mathbf{L}_f \) and solving for \( \omega_f \) yields

\[ \omega_f = \frac{m(l/4)}{I_{cm} + \frac{ml^2}{8}} v_0. \]  

(1.10.48)

If the stick is uniform, \( I_{cm} = ml^2/12 \) and Equation (1.10.48) reduces to

\[ \omega_f = \frac{6}{5} \frac{v_0}{l}. \]  

(1.10.49)

It may be tempting to try to calculate angular momentum about the contact point, where the putty hits the stick. If this is done, there is no initial angular momentum, and after the collision the angular momentum will be the sum of two parts, the angular momentum of the center of mass of the stick and the angular moment about the center of the stick,

\[ \mathbf{L}_f = \mathbf{r}_{\text{cm}} \times \mathbf{p}_{\text{cm}} + I_{cm} \omega_f. \]  

(1.10.50)

There are two crucial things to note: First, the speed of the center of mass is not the speed found in part b); the rotation must be included, so that

\[ v_{cm} = v_0 / 2 - \omega_f (l/4). \]
Second, the direction of \( \mathbf{r}_{cm} \times \mathbf{p}_{cm} \) with respect to the contact point is, from the right-hand rule, into the page, or the \(-\hat{k}\)-direction, opposite the direction of \( \omega_f \). This is to be expected, as the sum in Equation (1.10.50) must be zero. Adding the \( \hat{k}\)-components (the only components) in Equation (1.10.50),

\[
-\left(\frac{l}{2}\right)m\left(v_0 / 2 - \omega_f (l / 4)\right) + I_{cm} \omega_f = 0. \tag{1.10.51}
\]

Solving Equation (1.10.51) for \( \omega_f \) yields Equation (1.10.48).

This alternative derivation should serve two purposes. One is that it doesn’t matter which point we use to find angular momentum. The second is that use of foresight, in this case choosing the center of mass of the system so that the final velocity does not contribute to the angular momentum, can prevent extra calculation. It’s often a matter of trial and error (“learning by misadventure”) to find the “best” way to solve a problem.

e) The time of one rotation will be the same for all observers, independent of choice of origin. This fact is crucial in solving problems, in that the angular velocity will be the same (this was used in the alternate derivation for part d) above). The time for one rotation is the period \( T = 2\pi / \omega_f \) and the distance the center of mass moves is

\[
x_{cm} = v_{cm} T = 2\pi \frac{v_{cm}}{\omega_f} = 2\pi \frac{v_0 / 2}{\left(\frac{m(l / 4)}{I_{cm} + mI^2 / 8}\right)^{\frac{1}{2}}} v_0
\]

\[
= 2\pi \frac{I_{cm} + mI^2 / 8}{m(l / 2)}.
\]

Using \( I_{cm} = ml^2 / 12 \) for a uniform stick gives

\[
x_{cm} = \frac{5}{6} \pi l. \tag{1.10.53}
\]
Appendix 1.A Chasles’s Theorem: Rotation and Translation of a Rigid Body

We now return to our translating and rotating rod that we first considered when we began our discussion of rigid bodies. We shall now show that the motion of any rigid body consists of a translation of the center of mass and rotation about the center of mass.

We shall demonstrate this for a rigid body by dividing up the rigid body into point-like constituents. Consider two point-like constituents with masses $m_1$ and $m_2$. Choose a coordinate system with a choice of origin such that body 1 has position $\mathbf{r}_1$ and body 2 has position $\mathbf{r}_2$ (Figure 1.A.1). The relative position vector is given by

$$\mathbf{r}_{1,2} = \mathbf{r}_1 - \mathbf{r}_2$$  \hspace{1cm} (1.A.1)

![Figure 1.A.1](image-url) Two body coordinate system.

Recall we defined the center of mass vector, $\mathbf{R}_{cm}$, of the two-body system is as

$$\mathbf{R}_{cm} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$  \hspace{1cm} (1.A.2)

In the Figure 1.A.2 we show the center of mass coordinate system.
The position vector of the object 1 with respect to the center of mass is given by

\[ \mathbf{r}_{cm,1} = \mathbf{r}_1 - \mathbf{R}_{cm} = \mathbf{r}_1 - \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} = \frac{m_2}{m_1 + m_2} (\mathbf{r}_1 - \mathbf{r}_2) = \frac{\mu}{m_i} \mathbf{r}_{1,2} \quad (1.3) \]

where

\[ \mu = \frac{m_1 m_2}{m_1 + m_2} \quad (1.4) \]

is the reduced mass.

In addition, the relative position vector between the two objects is independent of the choice of reference frame,

\[ \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2 = (\mathbf{r}_{cm,1} + \mathbf{R}_{cm}) - (\mathbf{r}_{cm,2} + \mathbf{R}_{cm}) = \mathbf{r}_{cm,1} - \mathbf{r}_{cm,2} = \mathbf{r}_{cm,1,2} \quad (1.5) \]

Because the center of mass is at the origin in the center of mass reference frame,

\[ \frac{m_1 \mathbf{r}_{cm,1} + m_2 \mathbf{r}_{cm,2}}{m_1 + m_2} = \mathbf{0} \quad (1.6) \]

Therefore

\[ m_1 \mathbf{r}_{cm,1} = -m_2 \mathbf{r}_{cm,2} \quad (1.7) \]

or

\[ m_i \left| \mathbf{r}_{cm,i} \right| = m_j \left| \mathbf{r}_{cm,j} \right| \quad (1.8) \]

The displacement of object 1 about the center of mass is given by taking the derivative of Eq. (1.3),

\[ d\mathbf{r}_{cm,1} = \frac{\mu}{m_i} d\mathbf{r}_{1,2} \quad (1.9) \]
A similar calculation for the position of object 2 with respect to the center of mass yields
for the position and displacement with respect to the center of mass

$$\mathbf{r}_{cm,2} = \mathbf{r}_2 - \mathbf{R}_{cm} = -\frac{\mu}{m_2} \mathbf{r}_{1,2}$$  \hspace{1cm} (1.A.10)

and

$$d\mathbf{r}_{cm,2} = -\frac{\mu}{m_2} d\mathbf{r}_{1,2}.$$  \hspace{1cm} (1.A.11)

Let $i = 1, 2$. An arbitrary displacement of the $i^{th}$ object is given respectively by

$$d\mathbf{r}_i = d\mathbf{r}_{cm,i} + d\mathbf{R}_{cm}$$  \hspace{1cm} (1.A.12)

which is the sum of a displacement about the center of mass $d\mathbf{r}_{cm}$ and a displacement of
the center of mass $d\mathbf{R}_{cm}$.

The displacement of objects 1 and 2 are constrained by the condition that the distance
between the objects must remain constant since the body is rigid. In particular, the
distance between objects 1 and 2 is given by

$$\mathbf{r}_{1,2}^2 = (\mathbf{r}_1 - \mathbf{r}_2) \cdot (\mathbf{r}_1 - \mathbf{r}_2)$$  \hspace{1cm} (1.A.13)

Since this distance is constant we can differentiate Eq. (1.A.13), yielding the rigid body
condition that

$$0 = 2(\mathbf{r}_1 - \mathbf{r}_2) \cdot (d\mathbf{r}_1 - d\mathbf{r}_2) = 2\mathbf{r}_{1,2} \cdot d\mathbf{r}_{1,2}$$  \hspace{1cm} (1.A.14)

**Translation of the Center of Mass:**

This condition (Eq. (1.A.14)) can be satisfied if the relative displacement vector between
the two objects is zero,

$$d\mathbf{r}_{1,2} = d\mathbf{r}_1 - d\mathbf{r}_2 = \mathbf{0}$$  \hspace{1cm} (1.A.15)

This implies, using, Eq. (1.A.9) and Eq. (1.A.11), that the displacement with respect to
the center of mass is zero,

$$d\mathbf{r}_{cm,1} = d\mathbf{r}_{cm,2} = \mathbf{0}$$  \hspace{1cm} (1.A.16)
Thus by Eq. (1.A.12), the displacement of each object is equal to the displacement of the center of mass,

\[ d\mathbf{r}_i = d\mathbf{R}_{cm} \]  

(1.A.17)

which means that the body is undergoing pure translation.

**Rotation about the Center of Mass:**

Now suppose that \( d\mathbf{r}_{1,2} = d\mathbf{r}_1 - d\mathbf{r}_2 \neq \mathbf{0} \). The rigid body condition can be expressed in terms of the center of mass coordinates. Using Eq. (1.A.9), the rigid body condition (Eq. (1.A.14)) becomes

\[ 0 = 2 \frac{\mu}{m_1} \mathbf{r}_{i,2} \cdot d\mathbf{R}_{cm} \]  

(1.A.18)

Because the relative position vector between the two objects is independent of the choice of reference frame (Eq. (1.A.5)), the rigid body condition Eq. (1.A.14) in the center of mass reference frame is then given by

\[ 0 = 2 \mathbf{r}_{cm,1,2} \cdot d\mathbf{R}_{cm,1} \]  

(1.A.19)

This condition is satisfied if the relative displacement is perpendicular to the line passing through the center of mass,

\[ \mathbf{r}_{cm,1,2} \perp d\mathbf{R}_{cm,1} \]  

(1.A.20)

By a similar argument \( \mathbf{r}_{cm,1,2} \perp d\mathbf{R}_{cm,2} \). In order for these displacements to correspond to a rotation about the center of mass, the displacements must have the same angular displacement.

*Figure 1.A.3: infinitesimal angular displacements in the center of mass reference frame*
In Figure 1.A.3, the infinitesimal angular displacement of each object is given by

\[ d\theta_1 = \frac{d\vec{r}_{cm,1}}{|\vec{r}_{cm,1}|} \]  
(1.A.21)

\[ d\theta_2 = \frac{d\vec{r}_{cm,2}}{|\vec{r}_{cm,2}|} \]  
(1.A.22)

From Eq. (1.A.9) and Eq. (1.A.11), we can rewrite Eq. (1.A.22) as

\[ d\theta_1 = \frac{\mu}{m_1} \frac{d\vec{r}_{1,2}}{|\vec{r}_{cm,1}|} \]  
(1.A.23)

\[ d\theta_2 = \frac{\mu}{m_2} \frac{d\vec{r}_{1,2}}{|\vec{r}_{cm,2}|} \]  
(1.A.24)

Recall that in the center of mass reference frame \( m_1 |\vec{r}_{cm,1}| = m_2 |\vec{r}_{cm,2}| \) (Eq. (1.A.8)) and hence the angular displacements are equal,

\[ d\theta_1 = d\theta_2 = d\theta \]  
(1.A.25)

Therefore the displacement of the \( i^{th} \) object \( d\vec{r}_i \) differs from the displacement of the center of mass \( d\vec{R}_{cm} \) by a vector that corresponds to an infinitesimal rotation in the center of mass reference frame

\[ d\vec{r}_{cm,i} = d\vec{r}_i - d\vec{R}_{cm} \]  
(1.A.26)

So we have shown that the displacement of a rigid body is the vector sum of the displacement of the center of mass (translation of the center of mass) and an infinitesimal rotation about the center of mass.