Chapter 6 Appendices

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The Gravitational Field of a Spherical Shell of Matter.

Appendix 6.A:
Alternative Derivation of Acceleration for Uniform Circular Motion.

There are purely algebraic (as opposed to geometric) ways to find expressions for the velocity and acceleration of an object in a circular orbit. To do so, we will need to do a bit more calculus; that's the tradeoff.

Consider motion in a circle as described in Cartesian coordinates;

$$\vec{\mathbf{r}}(t) = R\left(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}\right). \tag{6.A.1}$$

For the current purpose, we will assume that R is a constant, but we will make no assumptions about $\theta(t)$ beyond the needed differentiability conditions. We then have

$$\begin{aligned} \vec{\mathbf{v}}(t) &= R\left(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\right)\frac{d\theta}{dt} \\ \left|\vec{\mathbf{v}}(t)\right| &= R\frac{d\theta}{dt} = R\omega. \end{aligned} \tag{6.A.2}$$

From these expressions, it is easily seen that

$$\vec{\mathbf{r}}\cdot\vec{\mathbf{v}} = R^2\omega\left(\cos\theta(t)\left(-\sin\theta(t)\right) + \sin\theta(t)\cos\theta(t)\right) = 0\,; \tag{6.A.3}$$

The position and velocity vectors are perpendicular, and hence the velocity, when nonzero, is tangent to the circle.

Finding the acceleration by taking another derivative will involve the product rule, with the result

$$\begin{aligned} \vec{\mathbf{a}} &= R\left(-\sin\theta(t)\,\hat{\mathbf{i}} + \cos\theta(t)\,\hat{\mathbf{j}}\right)\frac{d^2\theta}{dt^2} \\ &\quad -R\left(\cos\theta(t)\,\hat{\mathbf{i}} + \sin\theta(t)\,\hat{\mathbf{j}}\right)\left(\frac{d\theta}{dt}\right)^2 \\ &= \vec{\mathbf{v}}\frac{\alpha}{\omega} - \vec{\mathbf{r}}\omega^2. \end{aligned} \tag{6.A.4}$$

The first term in the last expression in (6.A.4) is the tangential acceleration vector, and is either parallel or antiparallel to the velocity vector. Since $|\vec{\mathbf{v}}| = R\omega$, $|a_\theta| = R|\alpha|$. If $\omega = 0$, the direction of the zero velocity vector is undetermined, and the first expression in (6.A.4) must be used; the tangential acceleration, if nonzero, will be tangent to the circle.

The second term in the last expression in (6.A.4) is the radial acceleration vector, and when not zero is in the $-\hat{\mathbf{r}}$-direction, radially inward.

In the above derivation, we used $\dfrac{d\,\hat{\mathbf{i}}}{dt} = \dfrac{d\,\hat{\mathbf{j}}}{dt} = \vec{\mathbf{0}}$ implicitly; that is, we assumed a nonrotating coordinate system. If, however, we choose to use cylindrical coordinates, the directions of the unit vectors change as the object moves around the circle (See Section 6.1 and Figure 6.1). To find the vector velocities, we need to consider the two vectors

$$\frac{d\hat{\mathbf{r}}}{dt} \text{ and } \frac{d\hat{\boldsymbol{\theta}}}{dt}. \tag{6.A.5}$$

What we will do first is to recognize that the changes in the unit vectors is a result of moving angularly; changing the distance from the origin does not change the direction of the unit vectors. We will use the chain rule to see that

$$\frac{d\hat{\mathbf{r}}}{dt} = \frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \text{ and } \frac{d\hat{\boldsymbol{\theta}}}{dt} = \frac{d\hat{\boldsymbol{\theta}}}{d\theta}\frac{d\theta}{dt}. \tag{6.A.6}$$

If this is new to you, the second expression in (6.A.6) may seem strange, in that it involves "differentiating theta with respect to theta." This is of course not the case; we are "differentiating theta-hat with respect to theta," that is, describing how the direction of a unit vector changes when we change the angular position. This is one of the reasons why we use different fonts, $\hat{\boldsymbol{\theta}}$ and θ, to describe the unit vector and the independent angular variable.

One way to find $\dfrac{d\hat{\mathbf{r}}}{d\theta}$ and $\dfrac{d\hat{\boldsymbol{\theta}}}{d\theta}$ is to proceed as above, expressing the unit vectors in Cartesian coordinates;

$$\begin{gathered} \hat{\mathbf{r}} = \cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}\,,\ \ \hat{\boldsymbol{\theta}} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} \\ \frac{d\hat{\mathbf{r}}}{d\theta} = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}} = \hat{\boldsymbol{\theta}}\,,\ \ \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\cos\theta\,\hat{\mathbf{i}} - \sin\theta\,\hat{\mathbf{j}} = -\hat{\mathbf{r}} \end{gathered} \tag{6.A.7}$$

Another way, one that does not rely on reference to a different coordinate system, is to consider that the only way for a unit vector to change is to change its direction. That is, we must have $\frac{d\hat{\mathbf{r}}}{d\theta} \perp \hat{\mathbf{r}}$ and $\frac{d\hat{\boldsymbol{\theta}}}{d\theta} \perp \hat{\boldsymbol{\theta}}$. Further, these being unit vectors, if the angle θ changes by a differential angle $d\theta$, the magnitude of the change in the unit vector is equal to the magnitude of the change in the angle;

$$\begin{gathered} \left|d\hat{\mathbf{r}}\right| = \left|d\theta\right| \quad \left|d\hat{\boldsymbol{\theta}}\right| = \left|d\theta\right| \\ \left|\frac{d\hat{\mathbf{r}}}{d\theta}\right| = \left|\frac{d\hat{\boldsymbol{\theta}}}{d\theta}\right| = 1 \end{gathered} \tag{6.A.8}$$

and so $\frac{d\hat{\mathbf{r}}}{d\theta}$ and $\frac{d\hat{\boldsymbol{\theta}}}{d\theta}$ are themselves unit vectors.

All that's left is to find directions. As the angle θ changes, the vector $\hat{\mathbf{r}}$ must change in this direction, $d\hat{\mathbf{r}} \parallel \hat{\boldsymbol{\theta}}$, and so $\frac{d\hat{\mathbf{r}}}{d\theta} = \hat{\boldsymbol{\theta}}$.

To find the direction of $\frac{d\hat{\boldsymbol{\theta}}}{d\theta}$, a similar geometric argument could be used, but there's a very elegant algebraic argument, one that does rely on results $\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}}$ and $\frac{d\hat{\mathbf{r}}}{d\theta}$. Starting from the orthogonality of the unit vectors,

$$\begin{aligned} \hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} &= 0 \\ \frac{d\hat{\mathbf{r}}}{d\theta} \cdot \hat{\boldsymbol{\theta}} + \hat{\mathbf{r}} \cdot \frac{d\hat{\boldsymbol{\theta}}}{d\theta} &= 0 \\ \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} + \hat{\mathbf{r}} \cdot \frac{d\hat{\boldsymbol{\theta}}}{d\theta} &= 0 \\ \hat{\mathbf{r}} \cdot \frac{d\hat{\boldsymbol{\theta}}}{d\theta} &= -1 \end{aligned} \tag{6.A.9}$$

and hence $\frac{d\hat{\mathbf{r}}}{d\theta}$ and $\hat{\mathbf{r}}$ are antiparallel; $\frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}}$.

The expressions we have found are so crucial that we will present them in a boxed set, along with two other expressions we have stated already;

$$\boxed{\begin{aligned}\frac{d\hat{\mathbf{r}}}{d\theta} &= \hat{\boldsymbol{\theta}}, \quad \frac{d\hat{\boldsymbol{\theta}}}{d\theta} = -\hat{\mathbf{r}} \\ \frac{d\hat{\mathbf{r}}}{dr} &= \vec{\mathbf{0}}, \quad \frac{d\hat{\boldsymbol{\theta}}}{dr} = \vec{\mathbf{0}}.\end{aligned}} \tag{6.A.10}$$

In fact, Equations (6.A.10) can be taken to define the coordinate system, but that's not part of our purpose.[1]

We are now ready to do the physics; with our preparation, this will be quick.

$$\begin{aligned}\vec{\mathbf{r}}(t) &= R\,\hat{\mathbf{r}} \\ \vec{\mathbf{v}}(t) &= \frac{d\vec{\mathbf{r}}}{dt} = R\frac{d\hat{\mathbf{r}}}{dt} = R\frac{d\hat{\mathbf{r}}}{d\theta}\frac{d\theta}{dt} \\ &= R\frac{d\theta}{dt}\hat{\boldsymbol{\theta}} \\ \vec{\mathbf{a}}(t) &= \frac{d\vec{\mathbf{v}}}{dt} = R\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + R\frac{d\theta}{dt}\frac{d\hat{\boldsymbol{\theta}}}{dt} = R\frac{d^2\theta}{dt^2}\hat{\boldsymbol{\theta}} + R\left(\frac{d\theta}{dt}\right)^2\frac{d\hat{\boldsymbol{\theta}}}{d\theta} \\ &= R\alpha\,\hat{\boldsymbol{\theta}} - R\omega^2\,\hat{\mathbf{r}}.\end{aligned} \tag{6.A.11}$$

We see that the acceleration does have both radial and tangential components, the radial component negative (centripetal acceleration always inward) and the tangential component either positive or negative, depending on the sign of α.

It should be noted that the expressions in (6.A.11) are only valid for circular motion; we assumed $dR/dt = 0$. Accounting for a changing distance from the origin leads to slightly more complicated expressions that will not be reproduced here.

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[1] More rigorously, and more generally, the derivatives in (6.A.10) should be *partial* derivatives, but when the unit vectors depend on only one independent variable, there is little need to make the distinction.
If we included the $\hat{\mathbf{k}}$ unit vector to account for three-dimensional cylindrical coordinates, all variations in $\hat{\mathbf{k}}$ and all derivatives with respect to z would vanish, so again we can use plain derivatives with respect to θ alone without much chance of ambiguity.
This would *not* be the case with spherical coordinates, however. For our purposes in this subject, we will consider these situations on a case-by-case basis.

Appendix 6.B: The Gravitational Field of a Spherical Shell of Matter

Consider a spherical shell of radius R with mass m_{S} that is uniformly distributed over the shell with mass per unit area $\sigma = \dfrac{m_{\text{S}}}{4\pi R^2}$.

In this appendix we will show the following two properties of the gravitational force that such a shell produces:

1) The gravitational force on a point-like object of mass m placed outside a spherical shell of matter of uniform surface mass density σ is the same force that would arise if all the mass of the shell were located at the center of the sphere.

2) The gravitational force on an object of mass m placed inside a spherical shell of matter is zero.

In summary,

$$\vec{\mathbf{F}}_{\text{object,S}}(r) = \begin{cases} -G\dfrac{m\, m_{\text{S}}}{r^2}\hat{\mathbf{r}}, & r > R \\ \vec{\mathbf{0}}, & r < R \end{cases} \tag{6.B.1}$$

where $\hat{\mathbf{r}}$ is the unit vector located at the position of the object and pointing radially away from the center of the shell.

We will use these properties to extend the results to the gravitational interaction between any two spherically symmetric bodies.

Any rigorous derivation of the above result will require use of spherical coordinates. This appendix will not go into the details of spherical coordinates, but rely on the geometry suggested by Figure 6.B.1 below. For a point on the surface of a sphere of radius $r = R$, the Cartesian coordinates are related to the spherical coordinates by

$$\begin{aligned} x &= R\sin\theta\cos\phi \\ y &= R\sin\theta\sin\phi \\ z &= R\cos\theta \end{aligned} \tag{6.B.2}$$

where $0 \le \theta \le \pi$ and $0 \le \phi \le 2\pi$.

You should be able to show quite easily that $x^2 + y^2 + z^2 = R^2$. You might also note that the angle θ in Figure 6.B.1 and Equations (6.B.2) is not the same as that in plane polar coordinates or cylindrical coordinates as shown in Figure 6.1 of the text. The relations in Figure 6.B.1 and Equations (6.B.2) are "Physics Notation," as opposed to what is done in most math subjects and textbooks. The age-old arguments for and against each system will not be presented here. However, keep in mind two good reasons for using physics notation:

- In most problems and derivations involving spherical symmetry, including this appendix, the angle θ will have a more prominent role than the angle ϕ and hence is given the more customary symbol for an angle.

- As shown in Figure 6.B.1, we still have at any point on the sphere a right-handed coordinate system, with $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\varphi}}$, directed counterclockwise when viewed from above (from the positive z-direction). (The symbol "$\boldsymbol{\varphi}$" is the bold form of "ϕ", and hence is used for the unit vector is the ϕ-direction.) Compare to $\hat{\mathbf{r}} \times \hat{\boldsymbol{\theta}} = \hat{\mathbf{k}}$ in cylindrical coordinates.

The angle θ is known as the *colatititude*, the complement of the latitude.

We can and will choose our z-axis to be directed from the center of the sphere to the position of the object, at position $z_0 \hat{\mathbf{k}}$, so that $z_0 \geq 0$ Let da be an infinitesimal surface area element on the shell and let dm_S be the infinitesimal mass element corresponding to da. The area element da is given by

$$da = (R\sin\theta d\phi)(R\,d\theta) = R^2 \sin\theta\, d\theta\, d\phi \;. \qquad (6.B.3)$$

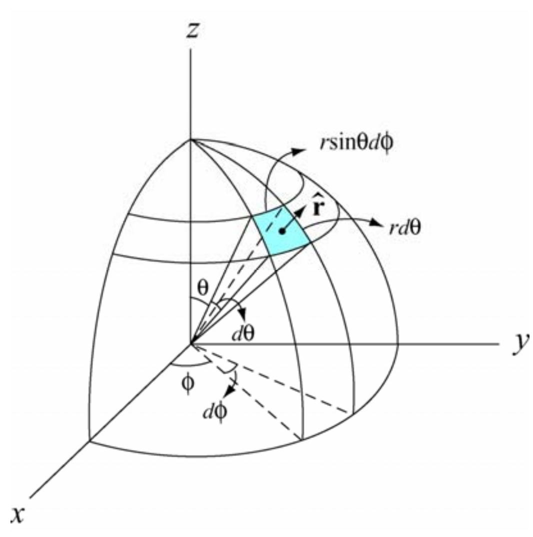

Figure 6.B.1: infinitesimal area element

Then the mass contained in that element is

$$dm_S = \sigma\, da = \sigma R^2 \sin\theta\; d\theta\; d\phi \,. \qquad (6.B.4)$$

The contribution from dm_{S} to the gravitational force on the object of mass m that lies outside the shell has a component pointing in the $\hat{\mathbf{k}}$-direction and a radial component pointing towards the z-axis. If the object is inside the shell, the $\hat{\mathbf{k}}$-component of the contribution to the force could be positive, negative or zero. By symmetry there is another mass element with the same differential mass dm_{S} on the other side of the shell with same colatitude θ but with ϕ replaced by $\phi \pm \pi$; this replacement changes the sign of x and y in Equations (6.B.2) but leaves z unchanged. This other mass element produces a gravitational force that exactly cancels the radial component of the force pointing towards the z-axis. The sum of the forces of these differential mass elements on the object has only a component in the $\hat{\mathbf{k}}$-direction. Therefore we want only

$$\left(d\vec{\mathbf{F}}_{\text{object, S}}\right)_z \equiv dF_z\,\hat{\mathbf{k}} = -G\frac{m\,dm_{\text{S}}}{s^2}\cos\alpha\,\hat{\mathbf{k}} \qquad (6.B.5)$$

where

$$s^2 = R^2 + z_0{}^2 - 2\,R\,z_0\cos\theta \quad \text{and} \quad \cos\alpha = \frac{z_0 - z'}{s} = \frac{z_0 - R\cos\theta}{s}. \qquad (6.B.6)$$

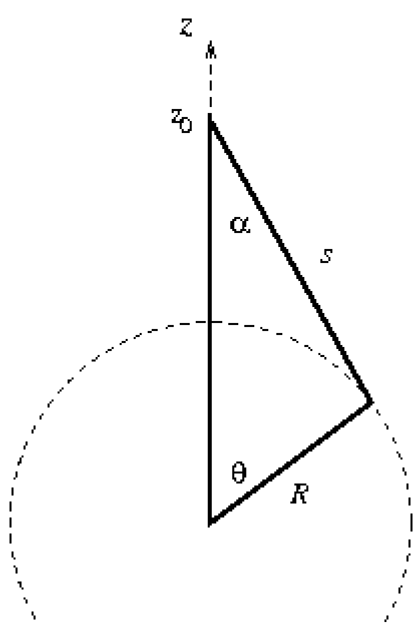

Figure 6.B.2: Geometry for calculating the force.

In Equation (6.B.6), s is the distance from any point on the ring to the position of the object, z' is the distance along the z-axis from the center of the shell to the plane of the differential ring and α is the angle that the vector of magnitude s from any point on the ring to the object makes with respect to the z-axis. The geometry is shown in Figure 6.B.2 above. The circle in the figure is a cross-section of the spherical shell.

Combining Equations (6.B.4), (6.B.5) and both expressions in (6.B.6),

$$dF_z = -G\frac{m\,dm_S}{s^2}\cos\alpha = -G\frac{m\,m_S}{4\pi R^2}\frac{R^2\sin\theta d\theta d\phi\left(z_0 - R\cos\theta\right)}{\left(R^2 + z_0^{\,2} - 2\,R\,z_0\cos\theta\right)^{3/2}}\,. \qquad (6.B.7)$$

The expression in Equation (6.B.7) can be simplified in preparation for integration. Make a change of variables by letting $u = z_0 - R\cos\theta$. Then $du = R\sin\theta\,d\theta$ and

$$s^2 = R^2 + z_0^{\,2} - 2\,R\,z_0\cos\theta = R^2 + 2u\,z_0 - z_0^{\,2}\,. \qquad (6.B.8)$$

Substitution of Equation (6.B.8) and the expression for u and du gives

$$dF_z = -G\frac{m\,m_s}{4\pi R^2}\frac{R\,u\,du\,d\phi}{\left(R^2 + 2u\,z_0 - z_0^{\,2}\right)^{3/2}}\,. \qquad (6.B.9)$$

When $\theta = 0$, $u = z_0 - R$; when $\theta = \pi$, $u = z_0 + R$. Thus the double integral becomes

$$F_z = \int\limits_{u=z-R}^{u=z+R}\int\limits_{\phi=0}^{\phi=2\pi} dF_z = -G\frac{m_s m}{4\pi R}\int\limits_{u=z-R}^{u=z+R}\int\limits_{\phi=0}^{\phi=2\pi}\frac{u\;du\;d\phi}{\left(R^2 + 2\;u\;z_0 - z_0^{\,2}\right)^{3/2}} \qquad (6.B.10)$$

We first integrate with respect to the ϕ-coordinate, contributing a multiplicative factor of 2π because the integrand is independent of ϕ. The double integral is then

$$F_z = -2\pi G\frac{m\,m_S}{4\pi R}\int\limits_{u=z-R}^{u=z+R}\frac{u\;du}{\left(R^2 + 2u\,z_0 - z_0^{\,2}\right)^{3/2}}\,. \qquad (6.B.11)$$

The above indefinite integral is not in everyone's toolkit, and not in many standard integral tables. (For those so inclined, a MAPLE worksheet that does the integral is given in the link at the end of this subsection.) One successful approach is to rewrite the numerator of the integrand as

$$u = \frac{R^2 + 2u\,z_0 - z_0^{\,2} - R^2 + z_0^{\,2}}{2\,z_0} = \frac{1}{2\,z_0}\left[\left(R^2 + 2u\,z_0 - z_0^{\,2}\right) + \left(z_0^{\,2} - R^2\right)\right]. \qquad (6.B.12)$$

The indefinite integral, apart from the leading constants, is then

$$\frac{1}{2z_0}\int\left[\frac{du}{\sqrt{R^2+2u\,z_0-z_0{}^2}}+\left(z_0{}^2-R^2\right)\frac{du}{\left(R^2+2u\,z_0-z_0{}^2\right)^{3/2}}\right]. \qquad (6.B.13)$$

These integrals are certainly in recognizable forms, leading to

$$\begin{aligned} F_z &= -G\frac{m\,m_S}{2R}\frac{1}{2z_0{}^2}\left(\sqrt{R^2+2u\,z_0-z_0{}^2}-\frac{\left(z_0{}^2-R^2\right)}{\sqrt{R^2+2u\,z_0-z_0{}^2}}\right)\Bigg|_{u=z_0-R}^{u=z_0+R} \\ &= -G\frac{m\,m_S}{2R}\frac{1}{2z_0{}^2}\left(\left(z_0+R-\frac{z_0{}^2-R^2}{z_0+R}\right)-\left(\sqrt{R^2-2\,z_0\,R+z_0{}^2}-\frac{\left(z_0{}^2-R^2\right)}{\sqrt{R^2-2\,z_0R+z_0{}^2}}\right)\right) \end{aligned} \qquad (6.B.14)$$

Now there is a subtlety. Since $\sqrt{R^2-2\,z_0\,R+z_0{}^2}$ is always positive, we have two special cases:

$$\sqrt{R^2-2\,z_0R+z_0{}^2}=\begin{cases} z_0-R\,, & z_0>R \\ R-z_0\,, & z_0<R. \end{cases} \qquad (6.B.15)$$

Then for $z_0>R$,

$$\begin{aligned} F_z &= -G\frac{m\,m_S}{2R}\frac{1}{2z_0{}^2}\left(\left(z_0+R-\frac{(z_0+R)(z_0-R)}{z_0+R}\right)-\left(z_0-R-\frac{(z_0+R)(z_0-R)}{z_0-R}\right)\right) \\ &= -G\frac{m\,m_S}{2R}\frac{1}{2z_0{}^2}4R=-G\frac{m\,m_S}{z_0{}^2}. \end{aligned} \qquad (6.B.16)$$

For $z_0<R$,

$$\begin{aligned} F_z &= -G\frac{m\,m_S}{2R}\frac{1}{2z_0{}^2}\left(\left(z_0+R-\frac{(z_0+R)(z_0-R)}{z_0+R}\right)-\left(R-z_0-\frac{(z_0+R)(z_0-R)}{R-z_0}\right)\right) \\ &= 0 \end{aligned} \qquad (6.B.17)$$

Collecting the results in Equations (6.B.16) and (6.B.17),

$$F_z=\begin{cases} -G\dfrac{m\,m_S}{z_0{}^2}, & z_0>R \\ 0\quad, & z_0<R, \end{cases} \qquad (6.B.18)$$

consistent with the stated goal as in (6.B.1).

A MAPLE worksheet that will perform the integral in Equation (6.B.11) (crude, but it works) may be downloaded here. Note that in this worksheet, the two cases must be considered separately, with the conditions entered in separate command lines.

This proves the result that the gravitational force inside the shell is zero and the gravitational force outside the shell is equivalent to locating all the mass of the object at the center of the shell. For some other spherically symmetric distribution, the attracting mass could be divided into concentric spherical shells and the above result applied to each shell. In general, this would involve an integral, and the limits of the integral would depend on whether or not the object is outside all of the layers or outside some and inside others.

If the object were itself a spherically symmetric body, not necessarily point-like, it can be seen that the same result holds, with z_0 replaced with the center of mass of the body, by considering that we have shown that the original attracting body behaves gravitationally like a point mass. From Newton's Third Law we could reverse the argument and say that the object behaves like a point mass. If the bodies overlap, as might be expected with gaseous spheres, this argument will not hold, but such bodies are not likely to be spherical.

Exercise Left to the Reader:

What if $z_0 = R$? For gravity, this is not likely these days (but some folks are thinking about it), but for E&M purposes it might be of concern.

Hint: Use $z_0 = R$ in both equations in (6.B.6), simplify as much as you can, and substitute into the integral in (6.B.5). Use the half-angle formulas

$$\begin{aligned} 1-\cos\theta &= 2\sin^2\left(\frac{\theta}{2}\right) \\ \sin\theta &= 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) \end{aligned} \tag{6.B.19}$$

and you're essentially done.

Alternate Method of Integration:

The integral in Equation (6.B.5), apart from multiplicative constants and the subsequent ϕ-integral, can be expressed

$$\int \frac{\cos\alpha \sin\theta}{s^2} d\theta, \tag{6.B.20}$$

where the θ-dependence of dm_S is given explicitly. What we will do is change the integration variable to s, and so we'll need relations between the angles α and θ, and the lengths s, z_0 and R.

Consider a triangle with sides of lengths s, z_0 and R, with α between the sides of lengths z_0 and s, and θ between the sides of lengths z_0 and R. (This is the triangle shown in Figure 6.A.2 above.) Using the law of cosines twice,

$$\begin{aligned} s^2 &= R^2 + z_0{}^2 - 2\,R\,z_0 \cos\theta \\ R^2 &= z_0{}^2 + s^2 - 2\,s\,z_0 \cos\alpha. \end{aligned} \tag{6.B.21}$$

Differentiating the first expression in (6.B.21), with R and z_0 constant,

$$2\,s\,ds = 2\,R\,z_0 \sin\theta\, d\theta, \tag{6.B.22}$$

and from the second expression in (6.B.21),

$$\cos\alpha = \frac{1}{2\,z_0}\left[\left(z_0{}^2 - R^2\right)\frac{1}{s} + s\right]. \tag{6.B.23}$$

We now have everything we need in terms of s. Substituting Equations (6.B.23), (6.B.22) and the first expression in (6.B.21) into (6.B.20), and using the limits for the definite integral for $z_0 > R$,

$$\begin{aligned} \int_{\theta=0}^{\theta=\pi} \frac{\cos\alpha\sin\theta}{s^2} d\theta &= \frac{1}{2\,z_0}\int_{z_0-R}^{z_0+R}\left[\left(z_0{}^2 - R^2\right)\frac{1}{s} + s\right]\frac{1}{s^2}\frac{s\,ds}{R\,z_0} \\ &= \frac{1}{2\,R\,z_0{}^2}\left[\left(z_0{}^2 - R^2\right)\int_{z_0-R}^{z_0+R}\frac{ds}{s^2} + \int_{z_0-R}^{z_0+R} ds\right]. \end{aligned} \tag{6.B.24}$$

No tables should be needed for these; the result is

$$\frac{1}{2Rz_0^{\ 2}}\left[\frac{\left(R^2-z_0^{\ 2}\right)}{s}+s\right]_{z_0-R}^{z_0+R} = \frac{1}{2Rz_0^{\ 2}}\left[\left(R-z_0\right)+\left(R+z_0\right)+2R\right] = \frac{2}{z_0^{\ 2}}, \tag{6.B.25}$$

the expected result when the multiplicative constants are included.

For $z_0 < R$, the lower limit is $R - z_0$, and the integral is

$$\frac{1}{2Rz_0^{\ 2}}\left[\frac{\left(R^2-z_0^{\ 2}\right)}{s}+s\right]_{R-z_0}^{z_0+R} = \frac{1}{2Rz_0^{\ 2}}\left[\left(R-z_0\right)-\left(R+z_0\right)+2z_0\right] = 0, \tag{6.B.26}$$

again the expected result.