Chapter 13 Rotational Dynamics

He sighed with the difficulty of talking mechanics to an unmechanical person. "There's a torque," he said. "It ain't balanced ---"

Any mechanic would have understood his drift at once. If a three-bladed propeller loses a blade, there are two blades left on one-third of its circumference, and nothing on the other two-thirds. All the resistance to its rotation under water is consequently concentrated upon one small section of the shaft, and a smooth revolution would be rendered impossible ...\(^1\)

C.S. Forester

_The African Queen_

**Introduction**

The physical objects that we encounter in the world consist of collections of atoms that are bound together to form systems of particles. When forces are applied, the shape of the body may be stretched or compressed like a spring, or sheared like jello. In some systems the constituent particles are very loosely bound to each other as in fluids and gasses, and the distances between the constituent particles will vary. We shall begin our study of extended objects by restricting ourselves to an ideal category of objects, rigid bodies, which do not stretch, compress, or shear.

A body is called a *rigid body* if the distance between any two points in the body does not change in time. Rigid bodies, unlike point masses, can have forces applied at different points in the body. For most objects, treating as a rigid body is an idealization, but a very good one. In addition to forces applied at points, forces may be distributed over the entire body. Distributed forces are difficult to analyze; however, for example, we regularly experience the effect of the gravitational force on bodies. Based on our experience observing the effect of the gravitational force on rigid bodies, we note that the gravitational force can be concentrated at a point in the rigid body called the *center of gravity*, which for small bodies (so that \(g\) may be taken as constant within the body) is identical to the *center of mass* of the body (we shall prove this fact in Appendix 13.A).

Let’s consider a rigid rod thrown in the air (Figure 13.1) so that the rod is spinning as its center of mass moves with velocity \(\vec{v}_m\). Rigid bodies, unlike point-like objects, can have forces applied at different points in the body. We have explored the physics of translational motion; now, we wish to investigate the properties of rotation exhibited in the rod’s motion, beginning with the notion that every particle is rotating about the center of mass with the same angular (rotational) velocity.

\(^1\) The authors of these notes suspect either a math error on Mr. Forester’s part or an oversight by his editors.
We can use Newton’s Second Law to predict how the center of mass will move. Since the only external force on the rod is the gravitational force (neglecting the action of air resistance), the center of mass of the body will move in a parabolic trajectory.

How was the rod induced to rotate? In order to spin the rod, we applied a torque with our fingers and wrist to one end of the rod as the rod was released. The applied torque is proportional to the angular acceleration. The constant of proportionality is called the \textit{moment of inertia}. When external forces and torques are present, the motion of a rigid body can be extremely complicated while it is translating and rotating in space. We shall begin our study of rotating objects by considering the simplest example of rigid body motion, rotation about a fixed axis.

\textbf{13.1 Fixed Axis Rotation: Rotational Kinematics}

\textbf{Fixed Axis Rotation}

When we studied static equilibrium, we demonstrated the need for two conditions: The total force acting on an object is zero, as is the total torque acting on the object. If the total torque is non-zero, then the object will start to rotate.

A simple example of rotation about a fixed axis is the motion of a compact disc in a CD player, which is driven by a motor inside the player. In a simplified model of this motion, the motor produces angular acceleration, causing the disc to spin. As the disc is set in motion, resistive forces oppose the motion until the disc no longer has any angular acceleration, and the disc now spins at a constant angular velocity. Throughout this process, the CD rotates about an axis passing through the center of the disc, and is perpendicular to the plane of the disc (see Figure 13.2). This type of motion is called \textit{fixed-axis rotation}. 

\textbf{Figure 13.1} The center of mass of a thrown rigid rod follows a parabolic trajectory while the rod rotates about the center of mass.
When we ride a bicycle forward, the wheels rotate about an axis passing through the center of each wheel and perpendicular to the plane of the wheel (Figure 13.3). As long as the bicycle does not turn, this axis keeps pointing in the same direction. This motion is more complicated than our spinning CD because the wheel is both moving (translating) with some center of mass velocity, $\mathbf{v}_{cm}$, and rotating.

**Figure 13.3** Fixed axis rotation and center of mass translation for a bicycle wheel.

When we turn the bicycle’s handlebars, we change the bike’s trajectory and the axis of rotation of each wheel changes direction. Other examples of non-fixed axis rotation are the motion of a spinning top, or a gyroscope, or even the change in the direction of the earth’s rotation axis. This type of motion is much harder to analyze, so we will restrict ourselves in this chapter to considering fixed axis rotation, with or without translation.

**Angular Velocity and Angular Acceleration**

When we considered the rotational motion of a point-like object in Chapter 6, we introduced an angle coordinate $\theta$, and then defined the angular velocity (Equation 6.2.7) as
\[ \omega \equiv \frac{d\theta}{dt}, \quad (13.1.1) \]

and angular acceleration (Equation 6.3.4) as

\[ \alpha \equiv \frac{d^2\theta}{dt^2}. \quad (13.1.2) \]

For a rigid body undergoing fixed-axis rotation, we can divide the body up into small volume elements with mass \( \Delta m_i \). Each of these volume elements is moving in a circle of radius \( r_{\perp,i} \) about the axis of rotation (Figure 13.4).

![Figure 13.4 Coordinate system for fixed-axis rotation.](image)

We will adopt the notation implied in Figure 13.4, and denote the vector from the axis to the point where the mass element is located as \( \mathbf{r}_{\perp,i} \), with \( r_{\perp,i} = |\mathbf{r}_{\perp,i}| \).

Because the body is rigid, all the volume elements will have the same angular velocity \( \omega \) and hence the same angular acceleration \( \alpha \). If the bodies did not have the same angular velocity, the volume elements would “catch up to” or “pass” each other, precluded by the rigid-body assumption.

**Sign Convention: Angular Velocity and Angular Acceleration**

Suppose we choose \( \theta \) to be increasing in the counterclockwise direction as shown in Figure 13.5.
Figure 13.5 Sign conventions for rotational motion.

If the rigid body rotates in the counterclockwise direction, then the angular velocity is positive, $\omega \equiv d\theta/dt > 0$. If the rigid body rotates in the clockwise direction, then the angular velocity is negative, $\omega \equiv d\theta/dt < 0$.

- If the rigid body *increases* its rate of rotation in the counterclockwise (positive) direction then the angular acceleration is positive, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt > 0$.
- If the rigid body *decreases* its rate of rotation in the counterclockwise (positive) direction then the angular acceleration is negative, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt < 0$.
- If the rigid body *increases* its rate of rotation in the clockwise (negative) direction then the angular acceleration is negative, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt < 0$.
- If the rigid body *decreases* its rate of rotation in the clockwise (negative) direction then the angular acceleration is positive, $\alpha \equiv d^2\theta/dt^2 = d\omega/dt > 0$.

To phrase this more generally, if $\alpha$ and $\omega$ have the same sign, the body is speeding up; if opposite signs, the body is slowing down. This general result is independent of the choice of positive direction of rotation.

Note that in Figure 13.2, the CD has a negative angular velocity as viewed from above; CDs do not operate the same way record player turntables do.

**Tangential Velocity and Tangential Acceleration**

Since the small volume $\Delta m_i$ element of mass is moving in a circle of radius $r_{L,i} = |\vec{r}_{L,i}|$ with angular velocity $\omega$, the element has a tangential velocity component

$$v_{\text{tan},i} = r_{L,i} \omega .$$  (13.1.3)
If the magnitude of the tangential velocity is changing, the volume element undergoes a tangential acceleration given by

\[ a_{\text{tan},i} = r_{\perp,i} \alpha. \]  

(13.1.4)

Recall from Chapter 6.3 Equation (6.3.14) that the volume element is always accelerating inward with magnitude

\[ \left| a_{\text{rad},i} \right| = \frac{v_{\text{tan},i}^2}{r_{\perp,i}} = r_{\perp,i} \omega^2. \]  

(13.1.5)

13.1.1 Example: Turntable, Part I

A turntable is a uniform disc of mass 1.2 kg and a radius 1.3 \times 10^1 cm. The turntable is spinning initially at a constant rate of \( f_0 = 33 \text{ cycles} \cdot \text{min}^{-1} \) (33 rpm). The motor is turned off and the turntable slows to a stop in 8.0 s. Assume that the angular acceleration is constant.

a) What is the initial angular velocity of the turntable?

b) What is the angular acceleration of the turntable?

Answer:

Initially, the disc is spinning with a frequency

\[ f_0 = \left(33 \frac{\text{cycles}}{\text{min}} \right) \left(1 \frac{\text{min}}{60 \text{ s}} \right) = 0.55 \text{ cycles} \cdot \text{s}^{-1} = 0.55 \text{ Hz}, \]  

(13.1.6)

so the initial angular velocity is

\[ \omega_0 = 2\pi f_0 = \left(2\pi \frac{\text{radian}}{\text{cycle}} \right) \left(0.55 \frac{\text{cycles}}{\text{s}} \right) = 3.5 \text{ rad} \cdot \text{s}^{-1}. \]  

(13.1.7)

The final angular velocity is zero, so the angular acceleration is

\[ \alpha = \frac{\Delta \omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} \left(\frac{-3.5 \text{ rad} \cdot \text{s}^{-1}}{8.0 \text{ s}} \right) = -4.3 \times 10^{-1} \text{ rad} \cdot \text{s}^{-2}. \]  

(13.1.8)

The angular acceleration is negative, and the disc is slowing down.
13.2 Torque

In order to understand the rotation of a rigid body we must introduce a new quantity, the torque. Let a force \( \vec{F}_p \) with magnitude \( F = |\vec{F}_p| \) act at a point \( P \). Let \( \vec{r}_{S,P} \) be the vector from the point \( S \) to a point \( P \), with magnitude \( r = |\vec{r}_{S,P}| \). The angle between the vectors \( \vec{r}_{S,P} \) and \( \vec{F}_p \) is \( \theta \) with \([0 \leq \theta \leq \pi]\) (Figure 13.6).

![Figure 13.6 Torque about a point \( S \) due to a force acting at a point \( P \)](image)

The torque about a point \( S \) due to force \( \vec{F}_p \) acting at \( P \), is defined by

\[
\vec{\tau}_S = \vec{r}_{S,P} \times \vec{F}_p. \tag{13.2.1}
\]

(See section 2.5 for a review of the definition of the cross product of two vectors). The magnitude of the torque about a point \( S \) due to force \( \vec{F}_p \) acting at \( P \), is given by

\[
\tau_S = r F \sin \theta. \tag{13.2.2}
\]

The SI units for torque are \([\text{N} \cdot \text{m}]\). The direction of the torque is perpendicular to the plane formed by the vectors \( \vec{r}_{S,P} \) and \( \vec{F}_p \) (for \([0 < \theta < \pi]\)), and by definition points in the direction of the unit normal vector to the plane \( \hat{n}_{RHR} \) as shown in Figure 13.7.

![Figure 13.7 Vector direction for the torque](image)
Recall that the magnitude of a cross product is the area of the parallelogram (the height times the base) defined by the two vectors. Figure 13.8 shows the two different ways of defining height and base for a parallelogram defined by the vectors \( \mathbf{r}_{s,p} \) and \( \mathbf{F}_p \).

![Figure 13.8 Area of the torque parallelogram.](image)

Let \( r_\perp = r \sin \theta \) and let \( F_\perp = F \sin \theta \) be the component of the force \( \mathbf{F}_p \) that is perpendicular to the line passing from the point \( S \) to \( P \). (Recall the angle \( \theta \) has a range of values \( 0 \leq \theta \leq \pi \) so both \( r_\perp \geq 0 \) and \( F_\perp \geq 0 \).) Then the area of parallelogram defined by \( \mathbf{r}_{s,p} \) and \( \mathbf{F}_p \) is given by

\[
\text{Area} = \tau_S = r_\perp F = r F_\perp = r F \sin \theta.
\]

(13.2.3)

We can interpret the quantity \( r_\perp \) as follows. We begin by drawing the line of action of the force \( \mathbf{F}_p \). This is a straight line passing through \( P \), parallel to the direction of the force \( \mathbf{F}_p \). Draw a perpendicular to this line of action that passes through the point \( S \) (Figure 13.9). The length of this perpendicular, \( r_\perp = r \sin \theta \), is called the moment arm about the point \( S \) of the force \( \mathbf{F}_p \).
Figure 13.9 The moment arm about the point $S$ associated with a force acting at the point $P$ is the perpendicular distance from $S$ to the line of action of the force passing through the point $P$.

You should keep in mind three important properties of torque:

1. The torque is zero if the vectors $\vec{r}_{S,p}$ and $\vec{F}_p$ are parallel ($\theta = 0$) or anti-parallel ($\theta = \pi$).

2. Torque is a vector whose direction and magnitude depend on the choice of a point $S$ about which the torque is calculated.

3. The direction of torque is perpendicular to the plane formed by the two vectors, $\vec{F}_p$ and $r = |\vec{r}_{S,p}|$ (the vector from the point $S$ to a point $P$).

Alternative Approach to Assigning a Sign Convention for Torque

In the case where all of the forces $\vec{F}_i$ and position vectors $\vec{r}_{i,p}$ are coplanar (or zero), we can, instead of referring to the direction of torque, assign a purely algebraic positive or negative sign to torque according to the following convention. We note that the arc in Figure 13.10a circles in counterclockwise direction. (Figures 13.10a and 13.10b use the simplifying assumption, for the purpose of the figure only, that the two vectors in question, $\vec{F}_p$ and $\vec{r}_{s,p}$ are perpendicular. The point $S$ about which torques are calculated is not shown.) We can associate with this counterclockwise orientation a unit normal vector $\hat{n}_{\text{RHR}}$ according to the right-hand rule: curl your right hand fingers in the counterclockwise direction and your right thumb will then point in the $\hat{n}_{\text{RHR}}$ direction. The arc in Figure 13.10b circles in the clockwise direction, and we associate this orientation with the unit normal $\hat{n}_{\text{LHR}}$. 
It’s important to note that the terms “clockwise” and “counterclockwise” might be different for different observers. For instance, if the plane containing $\vec{F}_p$ and $\vec{r}_{s,p}$ is horizontal, an observer above the plane and an observer below the plane would assign disagree on the two terms. For a vertical plane, the directions that two observers on opposite sides of the plane would be mirror images of each other, and so again the observers would disagree.

1. Suppose we choose counterclockwise as positive. Then we assign a positive sign to the torque when the torque is in the same direction as the unit normal $\hat{n}_{RHR}$, (Figure 13.10a).

2. Suppose we choose clockwise as positive. Then we assign a negative sign for the torque in Figure 13.10b since the torque is directed opposite to the unit normal $\hat{n}_{LHR}$.

With rare exceptions, these notes will take the counterclockwise direction to be positive.

13.3 Torque, Angular Acceleration, and Moment of Inertia

For fixed-axis rotation, there is a direct relation between the component of the torque along the axis of rotation and angular acceleration.

Consider the forces that act on the rotating body. Most generally, the forces on different volume elements will be different, and so we will denote the force on the volume element of mass $\Delta m_i$ by $\vec{F}_i$.

Choose the $z$-axis to lie along the axis of rotation. As in Section 13.1, divide the body into volume elements of mass $\Delta m_i$. Let the point $S$ denote a specific point along the axis of rotation (Figure 13.11). Each volume element undergoes a tangential acceleration as the volume element moves in a circular orbit of radius $r_{z,i} = |\vec{r}_{z,i}|$ about the fixed axis.
The vector from the point $S$ to the volume element is given by

$$\mathbf{r}_{S,i} = z_i \hat{k} + \mathbf{r}_{\bot,i} = z_i \hat{k} + r_{\bot,i} \hat{r}.$$  (13.3.1)

where $z_i$ is the distance along the axis of rotation between the point $S$ and the volume element. The torque about $S$ due to the force $\mathbf{F}_i$ acting on the volume element is given by

$$\tau_{S,i} = \mathbf{r}_{S,i} \times \mathbf{F}_i.$$  (13.3.2)

Substituting Equation (13.3.1) into Equation (13.3.2) gives

$$\tau_{S,i} = (z_i \hat{k} + r_{\bot,i} \hat{r}) \times \mathbf{F}_i.$$  (13.3.3)

For fixed-axis rotation, we are interested in the $z$-component of the torque, which must be the term

$$\tau_{S,i} = (r_{\bot,i} \hat{r} \times \mathbf{F}_i)_z.$$  (13.3.4)

since the cross product $z_i \hat{k} \times \mathbf{F}_i$ must be directed perpendicular to the plane formed by the vectors $\hat{k}$ and $\mathbf{F}_i$, hence perpendicular to the $z$-axis.

The total force acting on the volume element has components
\[ \mathbf{\tilde{F}}_i = F_{\text{radial},i} \hat{r} + F_{\text{tan},i} \hat{\theta} + F_{z,i} \hat{k}. \]  

(13.3.5)

The \( z \)-component \( F_{z,i} \) of the force cannot contribute a torque in the \( z \)-direction, and so substituting Equation (13.3.5) into Equation (13.3.4) yields

\[ \left( \tau_{s,i} \right)_z = \left( r_{\perp,i} \hat{r} \times \left( F_{\text{radial},i} \hat{r} + F_{\text{tan},i} \hat{\theta} \right) \right)_z. \]  

(13.3.6)

The radial force does not contribute to the torque about the \( z \)-axis, since

\[ r_{\perp,i} \hat{r} \times F_{\text{radial},i} \hat{r} = 0. \]  

(13.3.7)

So, we are interested in the contribution due to torque about the \( z \)-axis due to the tangential component of the force on the volume element (Figure 13.12). The component of the torque about the \( z \)-axis is given by

\[ \left( \tau_{s,i} \right)_z = \left( r_{\perp,i} \hat{r} \times F_{\text{tan},i} \hat{\theta} \right)_z = r_{\perp,i} F_{\text{tan},i} \hat{r}. \]  

(13.3.8)

The \( z \)-component of the torque is directed out of the page in Figure 13.12, where \( F_{\text{tan},i} \) is positive (the tangential force is directed counterclockwise, as in the figure).

\[ \text{Figure 13.12 Tangential force acting on a volume element.} \]

Applying Newton’s Second Law in the tangential direction,

\[ F_{\text{tan},i} = \Delta m_i a_{\text{tan},i}. \]  

(13.3.9)

Using the expression in (13.1.4) for tangential acceleration, we have that

\[ F_{\text{tan},i} = \Delta m_i r_{\perp,i} \alpha. \]  

(13.3.10)

From Equation (13.3.8), the component of the torque about the \( z \)-axis is then given by
\[
(r_{S,i})_z = r_{\perp,i} F_{\tan,i} = \Delta m_i (r_{\perp,i})^2 \alpha.
\] 

(13.3.11)

The total component of the torque about the \( z \)-axis is the summation of the torques on all the volume elements,

\[
(r_{S}^{\text{total}})_z = (r_{S,1})_z + (r_{S,2})_z + \cdots = \sum_{i=1}^{i=N} (r_{S,i})_z = \sum_{i=1}^{i=N} r_{\perp,i} F_{\tan,i}
\]

\[
= \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \alpha.
\] 

(13.3.12)

Since each element has the same angular acceleration, \( \alpha \), the summation becomes

\[
(r_{S}^{\text{total}})_z = \left( \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2 \right) \alpha.
\]

(13.3.13)

**Definition: Moment of Inertia about a Fixed Axis**

The quantity

\[
I_S = \sum_{i=1}^{i=N} \Delta m_i (r_{\perp,i})^2.
\]

(13.3.14)

is called the moment of inertia of the rigid body about a fixed axis passing through the point \( S \), and is a physical property of the body. The SI units for moment of inertia are \( \text{kg} \cdot \text{m}^2 \).

Thus Equation (13.3.13) shows that the \( z \)-component of the torque is proportional to the angular acceleration,

\[
(r_{S}^{\text{total}})_z = I_S \alpha,
\]

(13.3.15)

and the moment of inertia, \( I_S \), is the constant of proportionality.

This is very similar to Newton’s Second Law: the total force is proportional to the acceleration,

\[
\vec{F}^{\text{total}} = m^{\text{total}} \vec{a}.
\]

(13.3.16)

where the total mass, \( m^{\text{total}} \), is the constant of proportionality.
For a continuous mass distribution, the summation becomes an integral over the body

\[ I_s = \int_{\text{body}} dm (r_r)^2, \quad (13.3.17) \]

which will be explored in detail in the next section.

13.2.1 Example: Turntable, Part II

The turntable in Example 13.1.1, of mass 1.2 kg and radius 1.3×10^1 cm, has a moment of inertia \( I_s = 1.01 \times 10^{-2} \text{ kg}\cdot\text{m}^2 \) about an axis through the center of the disc and perpendicular to the disc. The turntable is spinning at an initial constant frequency of \( f_0 = 33 \text{ cycles}\cdot\text{min}^{-1} \). The motor is turned off and the turntable slows to a stop in 8.0 s due to frictional torque. Assume that the angular acceleration is constant. What is the magnitude of the frictional torque acting on the disc?

**Answer:**

We have already calculated the angular acceleration of the disc in Example 13.1.1, where we found that the angular acceleration is

\[ \alpha = \frac{\Delta \omega}{\Delta t} = \frac{\omega_f - \omega_0}{t_f - t_0} = \frac{-3.5 \text{ rad}\cdot\text{s}^{-1}}{8.0 \text{ s}} = -4.3 \times 10^{-1} \text{ rad}\cdot\text{s}^{-2} \quad (13.3.18) \]

and so the magnitude of the frictional torque is

\[ |\tau_{\text{friction}}| = I_s |\alpha| = (1.01 \times 10^{-2} \text{ kg}\cdot\text{m}^2)(4.3 \times 10^{-1} \text{ rad}\cdot\text{s}^{-2}) = 4.3 \times 10^{-3} \text{ N}\cdot\text{m}. \quad (13.3.19) \]

13.2.2 Example: Moment of Inertia of a Rod of Uniform Mass Density, Part I

Consider a thin uniform rod of length \( L \) and mass \( m \). In this problem, we will calculate the moment of inertia about an axis perpendicular to the rod that passes through the center of mass of the rod. A sketch of the rod, volume element, and axis is shown in Figure 13.13.

Choose Cartesian coordinates, with the origin at the center of mass of the rod, which is midway between the endpoints since the rod is uniform. Choose the \( x \)-axis to lie along the length of the rod, with the positive \( x \)-direction to the right, as in the figure.
Identify an infinitesimal mass element \( dm = \lambda \, dx \), located at a displacement \( x \) from the center of the rod, where the mass per unit length \( \lambda = m / L \) is a constant, as we have assumed the rod to be uniform.

When the rod rotates about an axis perpendicular to the rod that passes through the center of mass of the rod, the element traces out a circle of radius \( r_\perp = x \).

We add together the contributions from each infinitesimal element as we go from \( x = -L/2 \) to \( x = L/2 \). The integral is then

\[
I_{\text{cm}} = \int_{-L/2}^{L/2} (r_\perp)^2 \, dm = \lambda \int_{-L/2}^{L/2} (x^2) \, dx = \frac{\lambda}{3} \frac{x^3}{-L/2}^{L/2}
\]

\[
= \frac{m}{L} \frac{(L/2)^3}{3} - \frac{m}{L} \frac{(-L/2)^3}{3} = \frac{1}{12} mL^2.
\]

By using a constant mass per unit length along the rod, we need not consider variations in the mass density in any direction other than the \( x \)-axis. We also assume that the width is the rod is negligible. (Technically we should treat the rod as a cylinder or a rectangle in the \( x-y \) plane if the axis is along the \( z \)-axis. The calculation of the moment of inertia in these cases would be more complicated.)

### 13.4 Parallel Axis Theorem

Consider a rigid body of mass \( m \) undergoing fixed-axis rotation. Consider two parallel axes. The first axis passes through the center of mass of the body, and the moment of inertia about this first axis is \( I_{\text{cm}} \). The second axis passes through some other point \( S \) in the body. Let \( d_{s,\text{cm}} \) denote the perpendicular distance between the two parallel axes (Figure 13.14). Then the moment of inertia \( I_S \) about an axis passing through a point \( S \) is related to \( I_{\text{cm}} \) by

\[
I_S = I_{\text{cm}} + m \, d_{s,\text{cm}}^2.
\]
Figure 13.14 Geometry of the parallel axis theorem.

Proof of the Parallel Axis Theorem

Identify an infinitesimal volume element of mass \( dm \). The vector from the point \( S \) to the mass element is \( \vec{r}_{S,\text{dm}} \), the vector from the center of mass to the mass element is \( \vec{r}_{\text{cm,dm}} \), and the vector from the point \( S \) to the center of mass is \( \vec{r}_{S,\text{cm}} \). From Figure 13.14, we see that

\[
\vec{r}_{S,\text{dm}} = \vec{r}_{S,\text{cm}} + \vec{r}_{\text{cm,dm}}.
\]  

(13.4.2)

The notation gets complicated at this point. We are interested in distances from the respective axes, so denote the following vectors as motivated in Section 13.2:

- As in Figure 13.14 and Equation (13.4.2), \( \vec{r}_{\text{cm,dm}} \) is the vector from the center of mass to the position of the mass element of mass \( dm \). This vector has a component vector \( \vec{r}_{\text{cm,\|,dm}} \) parallel to the axis through the center of mass and a component vector \( \vec{r}_{\text{cm,\perp,dm}} \) perpendicular to the axis through the center of mass. The magnitude of the perpendicular component vector is

\[
|\vec{r}_{\text{cm,\perp,dm}}| = r_{\text{cm,\perp,dm}}.
\]  

(13.4.3)

- As in Figure 13.9 and Equation (13.4.2), \( \vec{r}_{S,\text{dm}} \) is the vector from the point \( S \) to the position of the mass element of mass \( dm \). This vector has a component vector \( \vec{r}_{S,\|,dm} \) parallel to the axis through the point \( S \) and a component vector
\( \mathbf{r}_{S,\perp,\text{dm}} \) perpendicular to the axis through the point \( S \). The magnitude of the perpendicular component vector is

\[
|\mathbf{r}_{S,\perp,\text{dm}}| = r_{S,\perp,\text{dm}}. \tag{13.4.4}
\]

- As in Figure 13.14 and Equation (13.4.2), \( \mathbf{r}_{S,\text{cm}} \) is the vector from the point \( S \) to the center of mass. This vector has a component vector \( \mathbf{r}_{S,\parallel,\text{cm}} \) parallel to both axes and a perpendicular component vector \( \mathbf{r}_{S,\perp,\text{cm}} \) of \( \mathbf{r}_{S,\text{cm}} \) perpendicular to both axes (the axes are parallel, of course). The magnitude of the perpendicular component vector is

\[
|\mathbf{r}_{S,\perp,\text{cm}}| = d_{S,\text{cm}}. \tag{13.4.5}
\]

Equation (13.4.2) is now expressed as two equations,

\[
\begin{align*}
\mathbf{r}_{S,\perp,\text{dm}} &= \mathbf{r}_{S,\perp,\text{cm}} + \mathbf{r}_{\text{cm},\perp,\text{dm}} \\
\mathbf{r}_{S,\parallel,\text{dm}} &= \mathbf{r}_{S,\parallel,\text{cm}} + \mathbf{r}_{\text{cm},\parallel,\text{dm}}.
\end{align*} \tag{13.4.6}
\]

At this point, note that if we had simply decided that the two parallel axes are parallel to the \( z \) -direction, we could have saved some steps and perhaps spared some of the notation with the triple subscripts. However, we want a more general result, one valid for cases where the axes are not fixed, or when different objects in the same problem have different axes. For example, consider the turning bicycle, for which the two wheel axes will not be parallel, or a spinning top that precesses (wobbles). Such cases will be considered in Chapter 16, and we will show the general case of the parallel axis theorem in anticipation of use for more general situations.

The moment of inertia about the point \( S \) is

\[
I_S = \int_{\text{body}} dm \left( r_{S,\perp,\text{dm}} \right)^2. \tag{13.4.7}
\]

From (13.4.6) we have

\[
\begin{align*}
\left( r_{S,\perp,\text{dm}} \right)^2 &= \mathbf{r}_{S,\perp,\text{dm}} \cdot \mathbf{r}_{S,\perp,\text{dm}} \\
&= \left( \mathbf{r}_{S,\perp,\text{cm}} + \mathbf{r}_{\text{cm},\perp,\text{dm}} \right) \cdot \left( \mathbf{r}_{S,\perp,\text{cm}} + \mathbf{r}_{\text{cm},\perp,\text{dm}} \right) \\
&= d_{S,\text{cm}}^2 + \left( r_{\text{cm},\perp,\text{dm}} \right)^2 + 2 \mathbf{r}_{S,\perp,\text{cm}} \cdot \mathbf{r}_{\text{cm},\perp,\text{dm}}. \tag{13.4.8}
\end{align*}
\]

Thus we have for the moment of inertia about \( S \),

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\[ I_S = \int \text{body} \, dm \left( d_{S, \text{cm}}^2 \right) + \int \text{body} \, dm \left( r_{\text{cm}, \perp \cdot dm} \right)^2 + 2 \int \text{body} \, dm \left( \mathbf{r}_{S, \perp \cdot \mathbf{r}_{\text{cm}, \perp \cdot dm}} \right). \]  

(13.4.9)

In the first integral in Equation (13.4.9), \( r_{S, \perp \cdot \text{cm}} = d_{S, \text{cm}} \) is the distance between the parallel axes and is a constant and may be taken out of the integral, and

\[ \int \text{body} \, dm \left( d_{S, \text{cm}}^2 \right) = m d_{S, \text{cm}}^2. \]  

(13.4.10)

The second term in Equation (13.4.9) is the moment of inertia about the axis through the center of mass,

\[ I_{\text{cm}} = \int \text{body} \, dm \left( r_{\text{cm}, \perp \cdot dm} \right)^2. \]  

(13.4.11)

The third integral in Equation (13.4.9) is zero. To see this, note that the term \( \mathbf{r}_{S, \perp \cdot \text{cm}} \) is a constant and may be taken out of the integral,

\[ 2 \int \text{body} \, dm \left( \mathbf{r}_{S, \perp \cdot \mathbf{r}_{\text{cm}, \perp \cdot dm}} \right) = \mathbf{r}_{S, \perp \cdot \mathbf{r}_{\text{cm}, \perp \cdot dm}} \cdot 2 \int \text{body} \, dm \left( \mathbf{r}_{\text{cm}, \perp \cdot dm} \right). \]  

(13.4.12)

The integral \( \int \text{body} \, dm \left( \mathbf{r}_{\text{cm}, \perp \cdot dm} \right) \) is the perpendicular component of the position of the center of mass with respect to the center of mass, and hence \( \mathbf{0} \), with the result that

\[ 2 \int \text{body} \, dm \left( \mathbf{r}_{S, \perp \cdot \mathbf{r}_{\text{cm}, \perp \cdot dm}} \right) = 0. \]  

(13.4.13)

Thus, the moment of inertia about \( S \) is just the sum of the first two integrals in Equation (13.4.9),

\[ I_S = I_{\text{cm}} + m d_{S, \text{cm}}^2. \]  

(13.4.14)

### 13.3.1 Example: Uniform Rod, Part II

Let point \( S \) be the left end of the rod of Example 13.2.1 and Figure 13.13. Then the distance from the center of mass to the end of the rod is \( d_{S, \text{cm}} = L/2 \). The moment of inertia \( I_S = I_{\text{end}} \) about an axis passing through the endpoint is related to the moment of
inertia about an axis passing through the center of mass, $I_{cm} = (1/12) m L^2$, according to Equation (13.4.14),

$$I_s = \frac{1}{12} mL^2 + \frac{1}{4} mL^2 = \frac{1}{3} mL^2.$$  \hspace{1cm} (13.4.15)

In this case it’s easy and useful to check by direct calculation. Use Equation (13.3.20) but with the limits changed to $x' = 0$ and $x' = L$, where $x' = x + L/2$;

$$I_{end} = \int_{body} r^2 \, dm = \lambda \int_0^L x'^2 \, dx'$$

$$= \lambda \frac{x'^3}{3} \bigg|_0^L = \frac{m (L)^3}{3} - \frac{m (0)^3}{3} = \frac{1}{3} mL^2.$$  \hspace{1cm} (13.4.16)

### 13.5 Simple Pendulum and Physical Pendulum

#### Simple Pendulum

A pendulum consists of an object hanging from the end of a string or rigid rod pivoted about the point $S$. The object is pulled to one side and allowed to oscillate. If the object has negligible size and the string or rod is massless, then the pendulum is called a *simple pendulum*. The force diagram for the simple pendulum is shown in Figure 13.15.

![Figure 13.15 A simple pendulum.](image-url)
The string or rod exerts no torque about the pivot point \( S \). The weight of the object has radial \( \hat{r} \)- and \( \hat{\theta} \)-components given by

\[
m\vec{g} = mg\left(\cos\theta \hat{r} - \sin\theta \hat{\theta}\right)\quad (13.5.1)
\]

and the torque about the pivot point \( S \) is given by

\[
\vec{\tau}_s = \vec{r}_{s,m} \times m\vec{g} = l \hat{r} \times m g \left(\cos\theta \hat{r} - \sin\theta \hat{\theta}\right) = -l m g \sin\theta \hat{k}\quad (13.5.2)
\]

and so the component of the torque in the \( z \)-direction (into the page in Figure 13.15 for \( \theta \) positive, out of the page for \( \theta \) negative) is

\[
(\tau_s)_z = -mgl \sin\theta.
\]

(13.5.3)

The moment of inertia of a point mass about the pivot point \( S \) is

\[
I_s = ml^2.\quad (13.5.4)
\]

From Equation (13.3.15) the rotational dynamical equation is

\[
(\tau_s)_z = I_s \alpha = I_s \frac{d^2\theta}{dt^2} = -mgl \sin\theta = ml^2 \frac{d^2\theta}{dt^2}.
\]

(13.5.5)

Thus we have the equation of motion for the simple pendulum,

\[
\frac{d^2\theta}{dt^2} = -\frac{g}{l} \sin\theta.\quad (13.5.6)
\]

When the angle of oscillation is small, then we can use the small angle approximation

\[
\sin\theta \cong \theta;
\]

(13.5.7)

the rotational dynamical equation for the pendulum becomes

\[
\frac{d^2\theta}{dt^2} \cong -\frac{g}{l} \theta.
\]

(13.5.8)
This equation is similar to the object-spring simple harmonic oscillator differential equation from Chapter 10.2, Equation 10.2.3,

\[ \frac{d^2x}{dt^2} = -\frac{k}{m} x, \quad (13.5.9) \]

which describes the oscillation of a mass about the equilibrium point of a spring. Recall that in Chapter 10.2, Equation 10.2.7, the angular frequency of oscillation was given by

\[ \omega_{\text{spring}} = \sqrt{\frac{k}{m}}. \quad (13.5.10) \]

By comparison, the frequency of oscillation for the pendulum is approximately

\[ \omega_{\text{pendulum}} \approx \sqrt{\frac{g}{l}}, \quad (13.5.11) \]

with period

\[ T = \frac{2\pi}{\omega_{\text{pendulum}}} \approx 2\pi \sqrt{\frac{l}{g}}. \quad (13.5.12) \]

A procedure for determining the period for larger angles is given in Appendix 13.B.

**Physical Pendulum**

A physical pendulum consists of a rigid body that undergoes fixed axis rotation about a fixed point \( S \) (Figure 13.16). The gravitational force acts at the center of mass of the physical pendulum (Appendix 13.A). Suppose the center of mass is a distance \( l_{cm} \) from the pivot point \( S \).
Figure 13.16 Physical pendulum.

The analysis is nearly identical to the simple pendulum. The torque about the pivot point is given by

\[ \vec{\tau}_S = \vec{r}_{S,\text{cm}} \times m \vec{g} = l_{\text{cm}} \vec{r} \times m \vec{g} \left( \cos \theta \, \hat{r} - \sin \theta \, \hat{\theta} \right) = -l_{\text{cm}} m g \sin \theta \, \hat{k}. \]  

(13.5.13)

Following the same steps that led from Equation (13.5.2) to Equation (13.5.6), the rotational dynamical equation for the physical pendulum is

\[ (\tau_S)_z = I_s \alpha = I_s \frac{d^2 \theta}{dt^2} - mgl_{\text{cm}} \sin \theta = I_s \frac{d^2 \theta}{dt^2}. \]  

(13.5.14)

Thus we have the equation of motion for the physical pendulum,

\[ \frac{d^2 \theta}{dt^2} = -\frac{mgl_{\text{cm}}}{I_s} \sin \theta. \]  

(13.5.15)

As with the simple pendulum, for small angles \( \sin \theta \approx \theta \) and Equation (13.5.15) reduces to the simple harmonic oscillator equation with angular frequency

\[ \omega_{\text{pendulum}} \approx \sqrt{\frac{mg}{I_s} l_{\text{cm}}} \]  

(13.5.16)

and period

\[ T_{\text{physical}} = \frac{2\pi}{\omega_{\text{pendulum}}} \approx 2\pi \sqrt{\frac{I_s}{mg l_{\text{cm}}}}. \]  

(13.5.17)

It is sometimes convenient to express the moment of inertia about the pivot point in terms of \( l_{\text{cm}} \) and \( I_{\text{cm}} \) using the parallel axis theorem in Equation (13.4.14), with \( d_{S,\text{cm}} = l_{\text{cm}} \), \( I_s = I_{\text{cm}} + m l_{\text{cm}}^2 \), with the result

\[ T_{\text{physical}} \approx 2\pi \sqrt{\frac{l_{\text{cm}} + I_{\text{cm}}}{g m l_{\text{cm}}}}. \]  

(13.5.18)
Thus, if the object is “small” in the sense that $I_{\text{cm}} \ll m l_{\text{cm}}^2$, the expressions for the physical pendulum reduce to those for the simple pendulum. Note that this is not the case shown in Figure 13.16.

### 13.6 Torque and Rotational Work

**Introduction**

When a constant torque $\tau_S$ is applied to an object, and the object rotates through an angle $\Delta \theta$ about an axis through the center of mass, then the torque does an amount of work $\Delta W = \tau_S \Delta \theta$ on the object. By extension of the linear work-energy theorem, the amount of work done is equal to the change in the rotational kinetic energy of the object,

$$W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_f^2 - \frac{1}{2} I_{\text{cm}} \omega_0^2 = K_{\text{rot},f} - K_{\text{rot},0}.$$  \hspace{0.7cm} (13.6.1)

The rate of doing this work is the rotational power exerted by the torque,

$$P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta W_{\text{rot}}}{\Delta t} = \tau_S \frac{d\theta}{dt} = \tau_S \omega.$$ \hspace{0.7cm} (13.6.2)

**Rotational work**

Consider a rigid body rotating about an axis. Each small element of mass $\Delta m_i$ in the rigid body is moving in a circle of radius $(r_{S,i})_{\perp}$ about the axis of rotation passing through the point $S$. Each mass element undergoes a small angular displacement $\Delta \theta$ under the action of a tangential force, $\mathbf{F}_{\text{tan},i} = F_{\text{tan},i} \hat{\mathbf{\theta}}$, where $\hat{\mathbf{\theta}}$ is the unit vector pointing in the tangential direction (Figure 13.7). The element will then have an associated displacement vector for this motion, $\Delta \mathbf{r}_{S,i} = (r_{S,i})_{\perp} \Delta \theta \hat{\mathbf{\theta}}$ and the work done by the tangential force is

$$\Delta W_i = \mathbf{F}_{\text{tan},i} \cdot \Delta \mathbf{r}_{S,i} = \left( F_{\text{tan},i} \hat{\mathbf{\theta}} \right) \cdot \left( (r_{S,i})_{\perp} \Delta \theta \hat{\mathbf{\theta}} \right) = F_{\text{tan},i} (r_{S,i})_{\perp} \Delta \theta.$$ \hspace{0.7cm} (13.6.3)

Applying Newton’s Second Law to the element $\Delta m_i$ in the tangential direction,

$$F_{\text{tan},i} = \Delta m_i a_{\text{tan},i}.$$ \hspace{0.7cm} (13.6.4)

Using the expression in Equation (13.1.4) for tangential acceleration we have that
\[ F_{\text{tan},i} = \Delta m_i \left( r_{S,i} \right)_\perp \alpha. \quad (13.6.5) \]

Thus the rotational work done on the mass element is
\[ \Delta W_i = \Delta m_i \left( r_{S,i} \right)_\perp^2 \alpha \Delta \theta. \quad (13.6.6) \]

Summing the rotational work done on all of the mass elements, we obtain
\[ \Delta W = \sum_i \Delta W_i = \left( \sum_i \Delta m_i \left( r_{S,i} \right)_\perp^2 \right) \alpha \Delta \theta. \quad (13.6.7) \]

In the limit that the discrete mass elements become infinitesimal continuous mass elements, \( \Delta m_i \to dm \), the summation becomes an integral over the body:
\[ \Delta W = \left( \sum_i \Delta m_i \left( r_{S,i} \right)_\perp^2 \right) \alpha \Delta \theta \to \left( \int_{\text{body}} dm \left( r_{S} \right)_\perp^2 \right) \alpha \Delta \theta. \quad (13.6.8) \]

Since the integral in this expression is just the moment of inertia about a fixed axis passing through the point \( S \), we have for the rotational work
\[ \Delta W = I_S \alpha \Delta \theta. \quad (13.6.9) \]

Since the \( z \)-component of the torque (in the direction along the axis of rotation) about \( S \) is given by
\[ (\tau_S)_z = I_S \alpha, \quad (13.6.10) \]
the rotational work is the product of the torque and the angular displacement,
\[ \Delta W = (\tau_S)_z \Delta \theta. \quad (13.6.11) \]

Recall the result of Equation (13.3.8) that the component of the torque (in the direction along the axis of rotation) about \( S \) due to the tangential force, \( F_{\text{tan},i} \), acting on the mass element \( \Delta m_i \) is
\[ (\tau_{S,i})_z = F_{\text{tan},i} \left( r_{S,i} \right)_\perp, \quad (13.6.12) \]
and the total torque is the sum

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\[(\tau_S)_z = \sum_i (\tau_{S,i})_z = \sum_i F_{\text{tan},i}(r_{S,i})_\perp\]  

(13.6.13)

and so the work done is

\[\Delta W = \sum_i \Delta W_i = \sum_i F_{\text{tan},i}(r_{S,i})_\perp \Delta \theta = (\tau_S)_z \Delta \theta .\]  

(13.6.14)

In the limit of small angles, \(\Delta \theta \rightarrow d\theta\), \(\Delta W \rightarrow dW\) and the differential rotational work is

\[dW = (\tau_S)_z d\theta .\]  

(13.6.15)

We can integrate this amount of rotational work as the angle coordinate of the rigid body changes from some initial value \(\theta = \theta_0\) to some final value \(\theta = \theta_f\),

\[W = \int dW = \int_{\theta_0}^{\theta_f} (\tau_S)_z d\theta .\]  

(13.6.16)

### Rotational Kinetic Energy

The general motion of a rigid body consists of a translation of the center of mass with velocity \(\mathbf{v}_{\text{cm}}\) and a rotation about the center of mass with angular velocity \(\omega_{\text{cm}}\).

Having defined translational kinetic energy in Chapter 7.2, Equation 7.2.1, we now define the rotational kinetic energy for a rigid body about its center of mass. Each individual mass element \(\Delta m_i\) undergoes circular motion about the center of mass with angular frequency \(\omega_{\text{cm}}\) in a circle of radius \((r_{\text{cm},i})_\perp\). Therefore the velocity of each element is given by \(\mathbf{v}_{\text{cm},i} = (r_{\text{cm},i})_\perp \omega_{\text{cm}} \hat{\theta}\). The rotational kinetic energy is then

\[K_{\text{cm},i} = \frac{1}{2} \Delta m_i \mathbf{v}_{\text{cm},i}^2 = \frac{1}{2} \Delta m_i (r_{\text{cm},i})_\perp^2 \omega_{\text{cm}}^2 .\]  

(13.6.17)

We now add up the kinetic energy for all the mass elements,

\[K_{\text{cm}} = \sum_i K_{\text{cm},i} = \left(\sum_i \frac{1}{2} \Delta m_i (r_{\text{cm},i})_\perp^2 \right) \omega_{\text{cm}}^2 = \left(\frac{1}{2} \int dm (r_{\text{cm}})_\perp^2 \right) \omega_{\text{cm}}^2 \]

(13.6.18)

\[= \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2 .\]

The total kinetic energy is the sum of the translational kinetic energy and the rotational kinetic energy,
\[ K_{\text{total}} = K_{\text{trans}} + K_{\text{rot}} = \frac{1}{2} m v_{\text{cm}}^2 + \frac{1}{2} I_{\text{cm}} \omega_{\text{cm}}^2. \quad (13.6.19) \]

The above assertion that the total kinetic energy consists of two parts, which is certainly plausible, is derived in Appendix 13.C.

**Rotational Work-Kinetic Energy Theorem**

We will now show that the rotational work is equal to the change in rotational kinetic energy. We begin by substituting our result from Equation (13.6.10) into Equation (13.6.15) for the infinitesimal rotational work,

\[ dW_{\text{rot}} = I_s \alpha \, d\theta. \quad (13.6.20) \]

Recall that the rate of change of angular velocity is equal to the angular acceleration, \( \alpha \equiv d\omega/dt \) and that the angular velocity is \( \omega \equiv d\theta/dt \). Note that in the limit of small displacements,

\[ \frac{d\omega}{dt} \, d\theta = \omega \cdot \frac{d\theta}{dt} = \omega \, \omega. \quad (13.6.21) \]

Therefore the infinitesimal rotational work is

\[ dW_{\text{rot}} = I_s \alpha \, d\theta = I_s \frac{d\omega}{dt} \, d\theta = I_s \omega \frac{d\theta}{dt} = I_s \omega \, \omega. \quad (13.6.22) \]

We can integrate this amount of rotational work as the angular velocity of the rigid body changes from some initial value \( \omega = \omega_0 \) to some final value \( \omega = \omega_f \),

\[ W_{\text{rot}} = \int dW_{\text{rot}} = \int_{\omega_0}^{\omega_f} I_s \omega \, \omega \, d\theta = \frac{1}{2} I_s \omega_f^2 - \frac{1}{2} I_s \omega_0^2. \quad (13.6.23) \]

When a rigid body is rotating about a fixed axis passing through a point \( S \) in the body, there is both rotation and translation about the center of mass unless \( S \) is the center of mass. If we choose the point \( S \) in the above equation for the rotational work to be the center of mass, then

\[ W_{\text{rot}} = \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},f}^2 - \frac{1}{2} I_{\text{cm}} \omega_{\text{cm},0}^2 = K_{\text{rot},f} - K_{\text{rot},0} = \Delta K_{\text{rot}}. \quad (13.6.24) \]

Recall the work-kinetic energy theorem stated that the total translational work done by all the forces is equal to the change in the translational kinetic energy of the center of mass,
\[ W_{\text{trans}} = \frac{1}{2} m v_{\text{cm},f}^2 - \frac{1}{2} m v_{\text{cm},0}^2 \equiv \Delta K_{\text{trans}}. \] (13.6.25)

Since we did not include the effect of rotational work in Equation (13.6.25), we now add the two contributions to the total work and find that the total work done on a rigid body is equal to the total change of the kinetic energy

\[ W_{\text{total}} = W_{\text{trans}} + W_{\text{rot}} = \left( \frac{1}{2} m v_{\text{cm},f}^2 - \frac{1}{2} m v_{\text{cm},0}^2 \right) + \left( \frac{1}{2} I_{\text{cm}} \omega_f^2 - \frac{1}{2} I_{\text{cm}} \omega_0^2 \right) = \Delta K_{\text{trans}} + \Delta K_{\text{rot}}. \] (13.6.26)

**Rotational Power**

The rotational power is defined as the rate of doing rotational work,

\[ P_{\text{rot}} \equiv \frac{dW_{\text{rot}}}{dt}. \] (13.6.27)

We can use our result for the infinitesimal work to find that the rotational power is the product of the applied torque with the angular velocity of the rigid body,

\[ P_{\text{rot}} = \frac{dW_{\text{rot}}}{dt} = (\tau_S)_z \frac{d \theta}{dt} = (\tau_S)_z \omega. \] (13.6.28)