# Chapter 17 The Kepler Problem: Planetary Mechanics and the Bohr Atom

## Kepler's Laws:[1]

- Each planet moves in an ellipse with the sun at one focus.
- The radius vector from the sun to a planet sweeps out equal areas in equal time.
- The period of revolution $T$ of a planet about the sun is related to the major axis $A$ of the ellipse by

$$T^2 = k\,A^3$$

where $k$ is the same for all planets.

## 17.1 Planetary Orbits: The Kepler Problem

### Introduction

Since Johannes Kepler first formulated the laws that describe planetary motion, scientists endeavored to solve for the equation of motion of the planets. In his honor, this problem has been named *The Kepler Problem*.

When there are more than two bodies, the problem becomes impossible to solve exactly. The most important "three-body problem" at the time involved finding the motion of the moon, since the moon interacts gravitationally with both the sun and the earth. Newton realized that if the exact position of the moon were known, the longitude of any observer on the earth could be determined by measuring the moon's position with respect to the stars.

In the eighteenth century, Leonhard Euler and other mathematicians spent many years trying to solve the three-body problem, and they raised a deeper question. Do the small contributions from the gravitational interactions of all the planets make the planetary system unstable over long periods of time? At the end of 18th century, Pierre Simon Laplace and others found a series solution to this stability question, but it was unknown whether or not the series solution converged after a long period of time. Henri Poincaré proved that the series actually diverged.

Poincaré went on to invent new mathematical methods that produced the modern fields of differential geometry and topology in order to answer the stability question using geometric arguments, rather than analytic methods. Poincaré and others did manage

[1] As stated in *An Introduction to Mechanics*, Daniel Kleppner and Robert Kolenkow, McGraw-Hill, 1973, p 401.

to show that the three-body problem was indeed stable, due to the existence of periodic solutions. Just as in the time of Newton and Leibniz and the invention of calculus, unsolved problems in celestial mechanics became the experimental laboratory for the discovery of new mathematics.

## 17.2 Reducing the Two-Body Problem into a One-Body Problem

We shall begin our solution of the two-body problem by showing how the motion of two bodies interacting via a gravitational force (two-body problem) is mathematically equivalent to the motion of a single body with a *reduced mass* given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \qquad (17.2.1)$$

that is acted on by an external central gravitational force. Once we solve for the motion of the reduced body in this *equivalent one-body problem*, we can then return to the real two-body problem and solve for the actual motion of the two original bodies.

The reduced mass was introduced in Section 10.7 of these notes. That section used similar but different notation from that used in this chapter.

Consider the gravitational force between two bodies with masses $m_1$ and $m_2$ as shown in Figure 17.1.

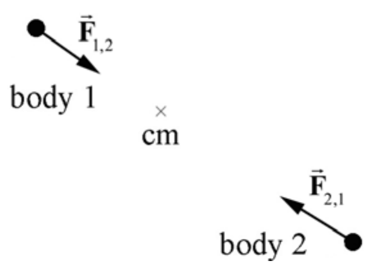


**Figure 17.1** Gravitational force between two bodies.

Choose a coordinate system with a choice of origin such that body 1 has position $\vec{\mathbf{r}}_1$ and body 2 has position $\vec{\mathbf{r}}_2$ (Figure 17.2). The *relative position vector* $\vec{\mathbf{r}}$ pointing from body 2 to body 1 is $\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2$. We denote the magnitude of $\vec{\mathbf{r}}$ by $|\vec{\mathbf{r}}| = r$, where $r$ is the distance between the bodies, and $\hat{\mathbf{r}}$ is the unit vector pointing from body 2 to body 1, so that

$$\vec{\mathbf{r}} = r\,\hat{\mathbf{r}} \qquad (17.2.2)$$

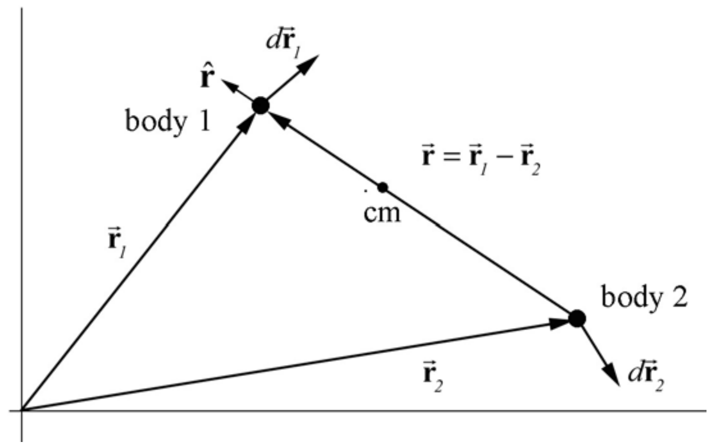


**Figure 17.2** Coordinate system for the two-body problem.

The force on body 1 (due to the interaction of the two bodies) can be described as

$$\vec{\mathbf{F}}_{1,2} = -F_{1,2}\,\hat{\mathbf{r}} = -G\frac{m_1 m_2}{r^2}\hat{\mathbf{r}}\,. \tag{17.2.3}$$

Recall that Newton's Third Law requires that the force on body 2 is equal in magnitude and opposite in direction to the force on body 1,

$$\vec{\mathbf{F}}_{1,2} = -\vec{\mathbf{F}}_{2,1}\,. \tag{17.2.4}$$

Newton's Second Law can be applied individually to the two bodies:

$$\vec{\mathbf{F}}_{1,2} = m_1\frac{d^2\vec{\mathbf{r}}_1}{dt^2}\,, \tag{17.2.5}$$

$$\vec{\mathbf{F}}_{2,1} = m_2\frac{d^2\vec{\mathbf{r}}_2}{dt^2}\,. \tag{17.2.6}$$

Dividing through by the mass in each of Equations (17.2.5) and (17.2.6) yields

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_1} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2}\,, \tag{17.2.7}$$

$$\frac{\vec{\mathbf{F}}_{2,1}}{m_2} = \frac{d^2\vec{\mathbf{r}}_2}{dt^2}\,. \tag{17.2.8}$$

Subtracting the expression in Equation (17.2.8) from that in Equation (17.2.7) gives

$$\frac{\vec{\mathbf{F}}_{1,2}}{m_1} - \frac{\vec{\mathbf{F}}_{2,1}}{m_2} = \frac{d^2\vec{\mathbf{r}}_1}{dt^2} - \frac{d^2\vec{\mathbf{r}}_2}{dt^2} = \frac{d^2\vec{\mathbf{r}}}{dt^2}. \quad (17.2.9)$$

Using Newton's Third Law as given in Equation (17.2.4), Equation (17.2.9) becomes

$$\vec{\mathbf{F}}_{1,2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) = \frac{d^2\vec{\mathbf{r}}}{dt^2}. \quad (17.2.10)$$

Using the *reduced mass* $\mu$ , as defined in Equation (17.2.1) ,

$$\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}, \quad (17.2.11)$$

Equation (17.2.10) becomes

$$\frac{\vec{\mathbf{F}}_{1,2}}{\mu} = \frac{d^2\vec{\mathbf{r}}}{dt^2}$$
$$\vec{\mathbf{F}}_{1,2} = \mu\frac{d^2\vec{\mathbf{r}}}{dt^2} \quad (17.2.12)$$

where $\vec{\mathbf{F}}_{1,2}$ is given by Equation (17.2.3).

Our result has a special interpretation using Newton's Second Law. Let $\mu$ be the reduced mass of a *reduced body* with position vector $\vec{\mathbf{r}} = r\,\hat{\mathbf{r}}$ with respect to an origin $O$, where $\hat{\mathbf{r}}$ is the unit vector pointing from the origin $O$ to the reduced body. Then the equation of motion, Equation (17.2.12), implies that the body of reduced mass $\mu$ is under the influence of an attractive gravitational force pointing toward the origin. So, the original two-body gravitational problem has now been reduced to an equivalent one-body problem, involving a reduced body with reduced mass $\mu$ under the influence of a central force $-\mathrm{F}_{1,2}\,\hat{\mathbf{r}}$. Note that in this reformulation, there is no body located at the central point (the origin $O$). The parameter $r$ in the two-body problem is the relative distance between the original two bodies, while the same parameter $r$ in the one-body problem is the distance between the reduced body and the central point.

## 17.3 Energy and Angular Momentum, Constants of the Motion

The equivalent one-body problem has two constants of the motion, energy $E$ and the angular momentum $L$ about the origin $O$. Energy is a constant because there are no external forces acting on the reduced body, and angular momentum is constant about the origin because the only force is directed towards the origin, and hence the torque about the origin due to that force is zero (the vector from the origin to the reduced body is anti-parallel to the force vector and $\sin\pi = 0$). Since angular momentum is constant, the orbit of the reduced body lies in a plane with the angular momentum vector pointing perpendicular to this plane.

In the plane of the orbit, choose polar coordinates $(r, \theta)$ for the reduced body (see Figure 17.3), where $r$ is the distance of the reduced body from the central point that is now taken as the origin, and $\theta$ is the angle that the reduced body makes with respect to a chosen direction, and which increases positively in the counterclockwise direction.

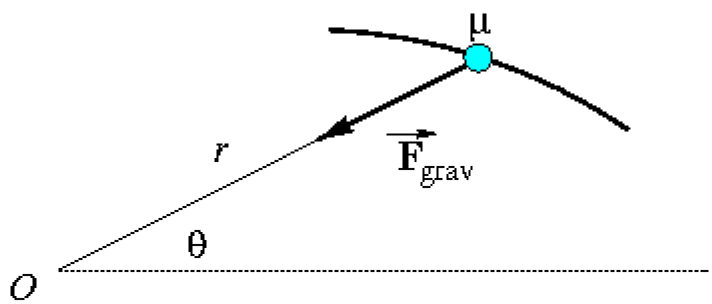


**Figure 17.3** Coordinate system for the orbit of the reduced body.

There are two approaches to describing the motion of the reduced body. We can try to find both the distance from the origin, $r(t)$ and the angle, $\theta(t)$, as functions of the parameter time, but in most cases explicit functions can't be found analytically. We can also find the distance from the origin, $r(\theta)$, as a function of the angle $\theta$. This second approach offers a spatial description of the motion of the reduced body (see Appendix 17.A).

### The Orbit Equation for the Reduced Body

Consider the reduced body with reduced mass given by Equation (17.2.1), orbiting about a central point under the influence of a radially attractive force given by Equation (17.2.3). Since the force is conservative, the potential energy with choice of zero reference point $U(\infty) = 0$ is given by

$$U(r) = -\frac{G m_1 m_2}{r}. \qquad (17.3.1)$$

The total energy $E$ is constant, and the sum of the kinetic energy and the potential energy is

$$E = \frac{1}{2}\mu v^2 - \frac{G\,m_1\,m_2}{r}\,. \qquad (17.3.2)$$

The kinetic energy term, $\mu v^2 / 2$, has the reduced mass and the relative speed $v$ of the two bodies. As in Chapters 5 and 7, we will use the notation

$$\begin{aligned} \vec{\mathbf{v}} &= v_{\text{rad}}\,\hat{\mathbf{r}} + v_{\tan}\hat{\boldsymbol{\theta}}, \\ v &= \left|\vec{\mathbf{v}}\right| = \left|\frac{d\vec{\mathbf{r}}}{dt}\right|, \end{aligned} \qquad (17.3.3)$$

where $v_{\text{rad}} = dr / dt$ and $v_{\tan} = r(d\theta / dt)$. Equation (17.3.2) then becomes

$$E = \frac{1}{2}\mu\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right] - \frac{G\,m_1\,m_2}{r}\,. \qquad (17.3.4)$$

The magnitude of the angular momentum with respect to the center of mass is

$$L = \mu\, r\, v_{\tan} = \mu\, r^2\,\frac{d\theta}{dt}\,. \qquad (17.3.5)$$

We shall explicitly eliminate the $\theta$ dependence from Equation (17.3.4) by using our expression in Equation (17.3.5),

$$\frac{d\theta}{dt} = \frac{L}{\mu\, r^2}\,. \qquad (17.3.6)$$

The mechanical energy as expressed in Equation (17.3.4) then becomes

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{1}{2}\frac{L^2}{\mu\, r^2} - \frac{G\,m_1\,m_2}{r}\,. \qquad (17.3.7)$$

Equation (17.3.7) is a separable differential equation involving the variable $r$ as a function of time $t$ and can be solved for the first derivative $dr / dt$,

$$\frac{dr}{dt} = \sqrt{\frac{2}{\mu}}\left(E - \frac{1}{2}\frac{L^2}{\mu\, r^2} + \frac{G\,m_1\,m_2}{r}\right)^{\frac{1}{2}}. \qquad (17.3.8)$$

Equation (17.3.8) can in principle be integrated directly for $r(t)$. In fact, in doing the integral no fewer than six cases need to be considered, and even then the solution is of the form $t(r)$ instead of $r(t)$. These integrals are presented in Appendix 17.E. The function

$r(t)$ can then, in principle, be substituted into Equation (17.3.6) and can then be integrated to find $\theta(t)$.

Instead of solving for the position of the reduced body as a function of time, we shall find a geometric description of the orbit by finding $r(\theta)$. We first divide Equation (17.3.6) by Equation (17.3.8) to obtain

$$\frac{d\theta}{dr} = \frac{\frac{d\theta}{dt}}{\frac{dr}{dt}} = \frac{\frac{L}{\mu r^2}}{\left(\frac{2}{\mu}\right)^{\frac{1}{2}}\left(E - \frac{1}{2}\frac{L^2}{\mu r^2} + \frac{G m_1 m_2}{r}\right)^{\frac{1}{2}}}. \tag{17.3.9}$$

The variables $r$ and $\theta$ are separable;

$$\begin{aligned} d\theta &= \frac{\frac{L}{\mu r^2}dr}{\left(\frac{2}{\mu}\right)^{\frac{1}{2}}\left(E - \frac{1}{2}\frac{L^2}{\mu r^2} + \frac{G m_1 m_2}{r}\right)^{\frac{1}{2}}} \\ &= \left(\frac{1}{2\mu}\right)^{\frac{1}{2}} \frac{\left(L/r^2\right)dr}{\left(E r^2 - \frac{1}{2}\frac{L^2}{\mu} + G m_1 m_2 r\right)^{\frac{1}{2}}}. \end{aligned} \tag{17.3.10}$$

Equation (17.3.10) can be integrated to find the radius as a function of the angle $\theta$; see Appendix 17.A for the exact integral solution. The result is called the *orbit equation* for the reduced body and is given by

$$r = \frac{r_0}{1 - \varepsilon\cos\theta} \tag{17.3.11}$$

where

$$r_0 = \frac{L^2}{\mu G m_1 m_2} \tag{17.3.12}$$

is a constant (known as the *semilatus rectum*) and

$$\varepsilon = \left(1 + \frac{2 E L^2}{\mu\left(G m_1 m_2\right)^2}\right)^{\frac{1}{2}} \tag{17.3.13}$$

is the *eccentricity* of the orbit. The two constants of the motion in terms of $r_0$ and $\varepsilon$ are

$$L = \left(\mu G m_1 m_2 r_0\right)^{\frac{1}{2}}$$
$$E = \frac{G m_1 m_2 \left(\varepsilon^2 - 1\right)}{2 r_0}. \tag{17.3.14}$$

An alternate derivation of Equation (17.3.11) is given in Appendix 17.F.

The orbit equation as given in Equation (17.3.11) is a general *conic section* and is perhaps somewhat more familiar in Cartesian coordinates. Let $x = r\cos\theta$ and $y = r\sin\theta$, with $r^2 = x^2 + y^2$. The orbit equation can be rewritten as

$$r = r_0 + \varepsilon r \cos\theta . \tag{17.3.15}$$

Using the Cartesian substitutions for $x$ and $y$, rewrite Equation (17.3.15) as

$$\left(x^2 + y^2\right)^{1/2} = r_0 + \varepsilon x . \tag{17.3.16}$$

Squaring both sides of Equation (17.3.16),

$$x^2 + y^2 = r_0^{\,2} + 2\varepsilon x r_0 + \varepsilon^2 x^2 . \tag{17.3.17}$$

After rearranging terms, Equation (17.3.17) is the general expression of a conic section with axis on the $x$-axis,

$$x^2\left(1-\varepsilon^2\right) - 2\varepsilon x r_0 + y^2 = r_0^{\,2} \tag{17.3.18}$$

(we now see that the dotted axis in Figure 17.3 can be taken to be the $x$-axis).

For a given $r_0 > 0$, corresponding to a given nonzero angular momentum as in Equation (17.3.11), there are four cases determined by the value of the eccentricity.

**Case 1:** When $\varepsilon = 0$, $E = E_{\min} < 0$ and $r = r_0$. Equation (17.3.18) is the equation for a circle,

$$x^2 + y^2 = r_0^{\,2} \tag{17.3.19}$$

**Case 2:** When $0 < \varepsilon < 1$, $E_{\min} < E < 0$ and Equation (17.3.18) describes an ellipse,

$$y^2 + A x^2 - B x = k \tag{17.3.20}$$

where $A > 0$ and $k$ is a positive constant. (Appendix 17.C shows how this expression may be expressed in the more traditional form involving the coordinates of the center of the ellipse and the semimajor and semiminor axes.)

**Case 3:** When $\varepsilon = 1$, $E = 0$ and Equation (17.3.18) describes a parabola,

$$x = \frac{y^2}{2r_0} - \frac{r_0}{2}. \tag{17.3.21}$$

**Case 4:** When $\varepsilon > 1$, $E > 0$ and Equation (17.3.18) describes a hyperbola,

$$y^2 - A x^2 - B x = k \tag{17.3.22}$$

where $A > 0$ and $k$ is a positive constant.

## 17.4 Energy Diagram, Effective Potential Energy, and Orbits of Motion

The energy (Equation (17.3.7)) of the reduced body moving in two dimensions can be reinterpreted as the energy of a reduced body moving in one dimension, the radial direction $r$, in an *effective potential energy* given by two terms,

$$U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}. \tag{17.4.1}$$

The total energy is still the same, but our interpretation has changed;

$$E = K_{\text{eff}} + U_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}, \tag{17.4.2}$$

where the *effective kinetic energy* $K_{\text{eff}}$ associated with the one-dimensional motion is

$$K_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2. \tag{17.4.3}$$

The graph of $U_{\text{eff}}$ as a function of $\mathsf{r} = r / r_0$, where $r_0$ as given in Equation (17.3.12), is shown in Figure 17.4. The upper curve (red, if you can see this in color) is proportional to $L^2 / \left(2\mu r^2\right) \sim 1/2r^2$. The lower blue curve is proportional to $-Gm_1 m_2 / r \sim -1/r$. The sum $U_{\text{eff}}$ is represented by the green curve. The minimum value of $U_{\text{eff}}$ is at $r = r_0$, as will be shown analytically below. The vertical scale is in units of $-U_{\text{eff}}\left(r_0\right)$.

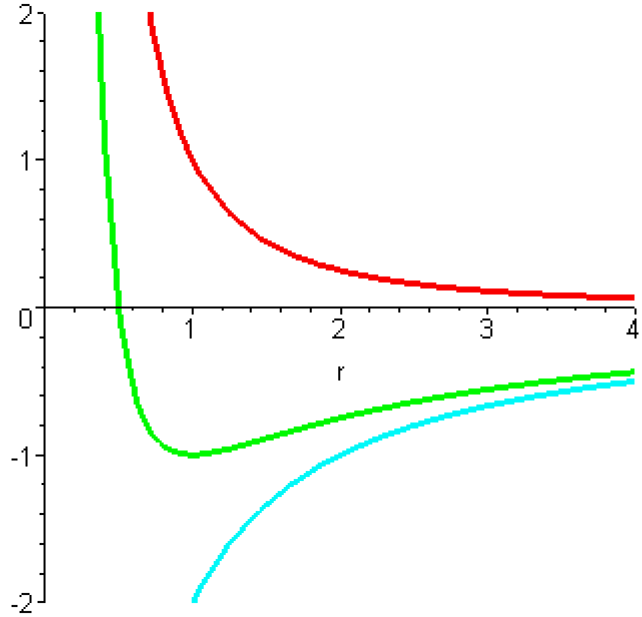


**Figure 17.4** Graph of effective potential energy.

Whenever the one-dimensional kinetic energy is zero, $K_{\text{eff}} = 0$, the energy is equal to the effective potential energy,

$$E = U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}. \tag{17.4.4}$$

Recall that the potential energy is defined to be the negative integral of the work done by the force. For our reduction to a one-body problem, using the effective potential, we will introduce an *effective force* such that

$$U_{\text{eff},B} - U_{\text{eff},A} = -\int_A^B \vec{\mathbf{F}}^{\text{eff}} \cdot d\vec{\mathbf{r}} = -\int_A^B F_r^{\text{eff}}\, dr \tag{17.4.5}$$

The fundamental theorem of calculus (for one variable) then states that the integral of the derivative of the effective potential energy function between two points is the effective potential energy difference between those two points,

$$U_{\text{eff},B} - U_{\text{eff},A} = \int_A^B \frac{dU_{\text{eff}}}{dr} dr \tag{17.4.6}$$

Comparing Equation (17.4.6) to Equation (17.4.5) shows that the radial component of the effective force is the negative of the derivative of the effective potential energy,

$$F_r^{\text{eff}} = -\frac{dU_{\text{eff}}}{dr} \qquad (17.4.7)$$

The effective potential energy describes the potential energy for a reduced body moving in one dimension. (Note that the effective potential energy is only a function of the variable $r$ and is independent of the variable $\theta$). There are two contributions to the effective potential energy, and the total radial component of the force is

$$F_r^{\text{eff}} = -\frac{d}{dr}U_{\text{eff}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r}\right) \qquad (17.4.8)$$

Thus there are two "forces" acting on the reduced body,

$$F_r^{\text{eff}} = F_{r,\ \text{centifugal}} + F_{r,\ \text{gravity}}\,, \qquad (17.4.9)$$

with an *effective centrifugal force* given by

$$F_{r,\text{centrifugal}} = -\frac{d}{dr}\left(\frac{L^2}{2\mu r^2}\right) = \frac{L^2}{\mu r^3} \qquad (17.4.10)$$

and the conventional gravitational force

$$F_{r,\ \text{gravity}} = -\frac{G m_1 m_2}{r^2}\,. \qquad (17.4.11)$$

With this nomenclature, let's review the four cases presented in Section 17.3.

**Case 1: Circular Orbit** $E = E_{\text{min}}$

The lowest energy state, $E_{\text{min}}$, corresponds to the minimum of the effective potential energy, $E_{\text{min}} = (U_{\text{eff}})_{\text{min}}$. When this condition is satisfied the effective kinetic energy is zero since $E = K_{\text{eff}} + U_{\text{eff}}$. The condition

$$K_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 = 0 \qquad (17.4.12)$$

implies that the radial velocity is zero, so the distance $r$ from the central point is a constant. This is the condition for a circular orbit. The condition for the minimum of the effective potential energy is

$$0 = \frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{\mu r^3} + \frac{G m_1 m_2}{r^2}\,. \qquad (17.4.13)$$

We can solve Equation (17.4.13) for $r$,

$$r \equiv r_0 = \frac{L^2}{G\,m_1 m_2}, \tag{17.4.14}$$

reproducing Equation (17.3.12).

**Case 2: Elliptic Orbit** $E_{\min} < E < 0$

When $K_{\text{eff}} = 0$, the mechanical energy is equal to the effective potential energy, $E = U_{\text{eff}}$, as in Equation (17.4.4). Having $dr/dt = 0$ corresponds to a point of closest or furthest approach as seen in Figure 17.4. This condition corresponds to the minimum and maximum values of $r$ for an elliptic orbit,

$$E = \frac{L^2}{2\mu r^2} - \frac{G\,m_1\,m_2}{r} \tag{17.4.15}$$

Equation (17.4.15) is a quadratic equation for the distance $r$,

$$r^2 + \frac{G\,m_1\,m_2}{E} r - \frac{L^2}{2\mu E} = 0 \tag{17.4.16}$$

with two roots

$$r = -\frac{G\,m_1\,m_2}{2E} \pm \left( \left( \frac{G\,m_1\,m_2}{2E} \right)^2 + \frac{L^2}{2\mu E} \right)^{1/2}. \tag{17.4.17}$$

Equation (17.4.17) may be simplified somewhat as

$$r = -\frac{G\,m_1\,m_2}{2E} \left( 1 \pm \left( 1 + \frac{2L^2 E}{\mu (G\,m_1 m_2)^2} \right)^{1/2} \right) \tag{17.4.18}$$

From Equation (17.3.13), the square root is the eccentricity $\varepsilon$,

$$\varepsilon = \left( 1 + \frac{2EL^2}{\mu \left( G\,m_1\,m_2 \right)^2} \right)^{\frac{1}{2}}, \tag{17.4.19}$$

and Equation (17.4.18) becomes

$$r = -\frac{G\,m_1\,m_2}{2E}\left(1 \pm \varepsilon\right). \tag{17.4.20}$$

A little algebra shows that

$$\begin{aligned}\frac{r_0}{1-\varepsilon^2} &= \frac{L^2 / \mu G\,m_1\,m_2}{1-\left(1+\dfrac{2L^2E}{\mu(G\,m_1\,m_2)^2}\right)} \\ &= \frac{L^2/\mu G m_1 m_2}{-2L^2E/\mu(Gm_1m_2)^2} \\ &= -\frac{G\,m_1\,m_2}{2E}.\end{aligned} \tag{17.4.21}$$

Substituting the last expression in (17.4.21) into Equation (17.4.20) gives an expression for the points of closest and furthest approach,

$$r = \frac{r_0}{1-\varepsilon^2}(1 \pm \varepsilon)\,. \tag{17.4.22}$$

The minus sign corresponds to the distance of closest approach,

$$r \equiv r_{\text{min}} = \frac{r_0}{1+\varepsilon} \tag{17.4.23}$$

and the plus sign corresponds to the distance of furthest approach,

$$r \equiv r_{\text{max}} = \frac{r_0}{1-\varepsilon}\,. \tag{17.4.24}$$

**Case 3: Parabolic Orbit** $E = 0$

The effective potential energy, as given in Equation (17.4.1), approaches zero ($U_{\text{eff}} \to 0$) when the distance $r$ approaches infinity ($r \to \infty$). Since the total energy is zero, when $r \to \infty$ the kinetic energy also approaches zero, $K_{\text{eff}} \to 0$. This corresponds to a parabolic orbit (see Equation (17.3.21)). Recall that in order for a body to escape from a planet, the body must have a total energy $E = 0$ (we set $U_{\text{eff}} = 0$ at infinity). This *escape velocity* condition corresponds to a parabolic orbit.

For a parabolic orbit, the body also has a distance of closest approach. This distance $r_{\text{par}}$ can be found from the condition

$$E = U_{\text{eff}} = \frac{L^2}{2\mu r^2} - \frac{G m_1 m_2}{r} = 0 \,. \qquad (17.4.25)$$

Solving Equation (17.4.25) for $r$ yields

$$r_{\text{par}} = \frac{L^2}{2\mu G m_1 m_2} = \frac{1}{2} r_0 \,; \qquad (17.4.26)$$

the fact that the minimum distance to the origin (the *focus* of a parabola) is half the semilatus rectum is a well-known property of a parabola.

**Case 4: Hyperbolic Orbit** $E > 0$

When $E > 0$, in the limit as $r \to \infty$ the kinetic energy is positive, $K_{\text{eff}} > 0$. This corresponds to a hyperbolic orbit (see Equation (17.3.22)). The condition for closest approach is similar to Equation (17.4.15) except that the energy is now positive. This implies that there is only one positive solution to the quadratic Equation (17.4.16), the distance of closest approach for the hyperbolic orbit

$$r_{\text{hyp}} = \frac{r_0}{1 + \varepsilon} \,. \qquad (17.4.27)$$

The constant $r_0$ is independent of the energy and from Equation (17.3.13) as the energy of the reduced body increases, the eccentricity increases, and hence from Equation (17.4.27), the distance of closest approach gets smaller.

## 17.5 Orbits of the Two Bodies

The orbit of the reduced body can be circular, elliptical, parabolic or hyperbolic, depending on the values of the two constants of the motion, the angular momentum and the energy. Once we have the explicit solution (in this discussion, $r(\theta)$) for the reduced body, we can find the actual orbits of the two bodies.

Choose a coordinate system as we did for the reduction of the two-body problem (Figure 17.5).

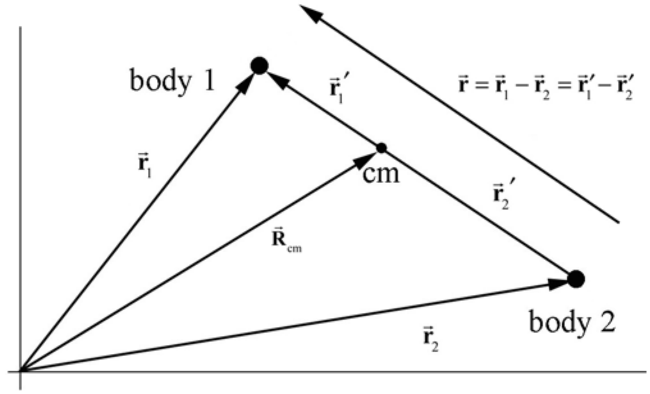


**Figure 17.5** Center of mass coordinate system.

The center of mass of the system is given by

$$\vec{\mathbf{R}}_{\text{cm}} = \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2} . \tag{17.5.1}$$

Let $\vec{\mathbf{r}}_1'$ be the vector from the center of mass to body 1 and $\vec{\mathbf{r}}_2'$ be the vector from the center of mass to body 2. Then, by the geometry in Figure 17.5,

$$\vec{\mathbf{r}} = \vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_1' - \vec{\mathbf{r}}_2' \tag{17.5.2}$$

and hence

$$\vec{\mathbf{r}}_1' = \vec{\mathbf{r}}_1 - \vec{\mathbf{R}}_{\text{cm}} = \vec{\mathbf{r}}_1 - \frac{m_1 \vec{\mathbf{r}}_1 + m_2 \vec{\mathbf{r}}_2}{m_1 + m_2} = \frac{m_2 \left(\vec{\mathbf{r}}_1 - \vec{\mathbf{r}}_2\right)}{m_1 + m_2} = \frac{\mu}{m_1} \vec{\mathbf{r}} . \tag{17.5.3}$$

A similar calculation shows that

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2} \vec{\mathbf{r}} . \tag{17.5.4}$$

Thus each body undergoes a motion about the center of mass in the same manner that the reduced body moves about the central point given by Equation (17.3.11). The only difference is that the distance from either body to the center of mass is shortened by a factor $\mu / m_i$. When the orbit of the reduced body is an ellipse, then the orbits of the two bodies are also ellipses, as shown in Figure 17.6.

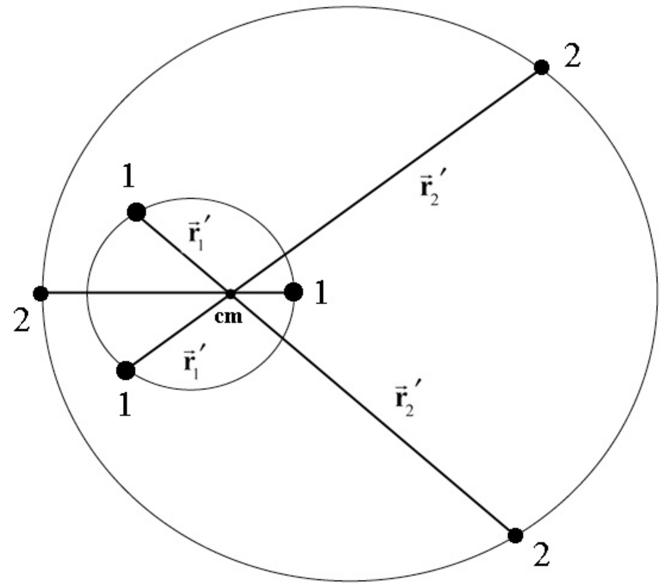


**Figure 17.6** The elliptical motion of bodies under mutual gravitation.

When one mass is much smaller than the other, for example $m_1 \ll m_2$, then the reduced mass is approximately the smaller mass,

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \cong \frac{m_1 m_2}{m_2} = m_1 \tag{17.5.5}$$

The center of mass is located approximately at the position of the larger mass, body 2 of mass $m_2$. Thus body 1 moves according to

$$\vec{\mathbf{r}}_1' = \frac{\mu}{m_1}\vec{\mathbf{r}} \cong \vec{\mathbf{r}} \tag{17.5.6}$$

and body 2 is approximately stationary,

$$\vec{\mathbf{r}}_2' = -\frac{\mu}{m_2}\vec{\mathbf{r}} \cong \frac{m_1}{m_2}\vec{\mathbf{r}} \cong \vec{\mathbf{0}} \tag{17.5.7}$$

## 17.6 Kepler's Laws

### Elliptic Orbit Law

*Each planet moves in an ellipse with the sun at one focus.*

When the energy is negative, $E < 0$, and according to Equation (17.3.13),

$$\varepsilon = \left(1 + \frac{2 E L^2}{\mu (G m_1 m_2)^2}\right)^{\frac{1}{2}} \tag{17.6.1}$$

and the eccentricity must fall within the range $0 \le \varepsilon < 1$. These orbits are either circles or ellipses. Note the elliptic orbit law is only valid if we assume that there is only one central force acting. We are ignoring the gravitational interactions due to all the other bodies in the universe, a necessary approximation for our analytic solution.

**Equal Area Law**

*The radius vector from the sun to a planet sweeps out equal areas in equal time.*

Using analytic geometry, the sum of the areas of the triangles in Figure 17.7 is given by

$$\Delta A = \frac{1}{2}(r\,\Delta\theta)r + \frac{(r\,\Delta\theta)}{2}\Delta r = \frac{1}{2}(r\,\Delta\theta)r + \frac{(r\,\Delta\theta)}{2}\Delta r \tag{17.6.2}$$

in the limit of small $\Delta\theta$ (the area of the small piece on the right, bounded on one side by the circular segment, is approximated by that of a triangle).

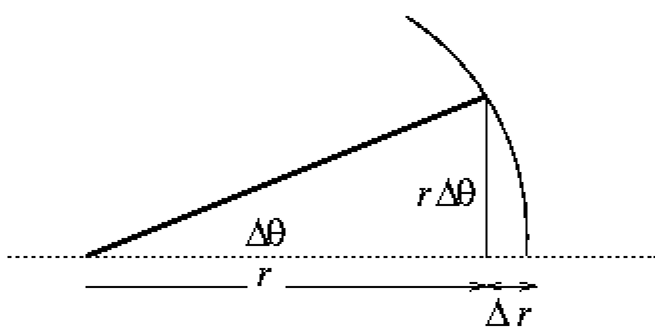


**Figure** 17.**7** Kepler's equal area law.

The average rate of the change of area, $\Delta A$, in time $\Delta t$, is given by

$$\frac{\Delta A}{\Delta t} = \frac{1}{2}\frac{(r\,\Delta\theta)r}{\Delta t} + \frac{(r\Delta\theta)}{2}\frac{\Delta r}{\Delta t}. \tag{17.6.3}$$

In the limit as $\Delta t \to 0$, $\Delta\theta \to 0$, this becomes

$$\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt}. \tag{17.6.4}$$

Note that in this approximation, we are essentially neglecting the small piece on the right in Figure 17.7

Recall that according to Equation (17.3.6) (reproduced below as Equation (17.6.5)), the angular momentum is related to the angular velocity $d\theta / dt$ by

$$\frac{d\theta}{dt} = \frac{L}{\mu r^2} \qquad (17.6.5)$$

and Equation (17.6.4) is then

$$\frac{dA}{dt} = \frac{L}{2\mu}. \qquad (17.6.6)$$

Since $L$ and $\mu$ are constants, the rate of change of area with respect to time is a constant. This is often familiarly referred to by the expression: *equal areas are swept out in equal times* (see Kepler's Laws at the beginning of this chapter).

**Period Law**

> *The period of revolution $T$ of a planet about the sun is related to the major axis $A$ of the ellipse by*
>
> $$T^2 = k A^3$$
>
> *where $k$ is the same for all planets.*

When Kepler stated his period law for planetary orbits based on observation, he only noted the dependence on the larger mass of the sun. Since the mass of the sun is much greater than the mass of the planets, his observation is an excellent approximation.

Equation (17.6.6) can be rewritten in the form

$$2\mu \frac{dA}{dt} = L. \qquad (17.6.7)$$

Equation (17.6.7) can be integrated as

$$\int_{\text{orbit}} 2\mu\, dA = \int_0^T L\, dt \qquad (17.6.8)$$

where $T$ is the period of the orbit. For an ellipse,

$$\text{Area} = \int_{\text{orbit}} dA = \pi\, ab \qquad (17.6.9)$$

where $a$ is the semimajor axis and $b$ is the semiminor axis. (Appendix 17.D derives this result from Equation (17.3.11).)

Thus we have

$$T = \frac{2\mu\,\pi\, ab}{L}. \qquad (17.6.10)$$

Squaring Equation (17.6.10) then yields

$$T^2 = \frac{4\pi^2\mu^2 a^2 b^2}{L^2}. \qquad (17.6.11)$$

In Appendix 17.B, the angular momentum is given in terms of the semimajor axis and the eccentricity by Equation (B.1.10). Substitution for the angular momentum into Equation (17.6.11) yields

$$T^2 = \frac{4\pi^2\mu^2 a^2 b^2}{\mu G m_1 m_2\, a\left(1-\varepsilon^2\right)}. \qquad (17.6.12)$$

In Appendix 17.B, the semi-minor axis is given by Equation (B.3.7), which upon substitution into Equation (17.6.12) yields

$$T^2 = \frac{4\pi^2\mu^2 a^3}{\mu G m_1 m_2}. \qquad (17.6.13)$$

Using Equation (17.2.1) for reduced mass, the square of the period of the orbit is proportional to the semi-major axis cubed,

$$T^2 = \frac{4\pi^2 a^3}{G\left(m_1 + m_2\right)}. \qquad (17.6.14)$$

## 17.7 The Bohr Atom

Numerical values of physical constants are from the Particle Data Group tables, available from http://pdg.lbl.gov/2006/reviews/consrpp.pdf.

Consider the electric force between two pointlike objects with charges $q_1$ and $q_2$. The force law is an inverse square law, like the gravitational force. The difference is that the constant $-G\,m_1 m_2$ is replaced by $k\,q_1 q_2$ where

$$k = \frac{1}{4\pi\varepsilon_0} = 8.987551788 \times 10^9 \ \mathrm{N \cdot m^2 \cdot C^{-2}} \tag{17.7.1}$$

(the constant "$k$" is not a spring constant or the Boltzmann constant, merely a reflection of our finite alphabet).

The minus sign in the gravitational interaction does not appear in the electric interaction because there are two types of electric charge, positive and negative. The electric force is attractive for charges of opposite sign and repulsive for charges of the same sign.

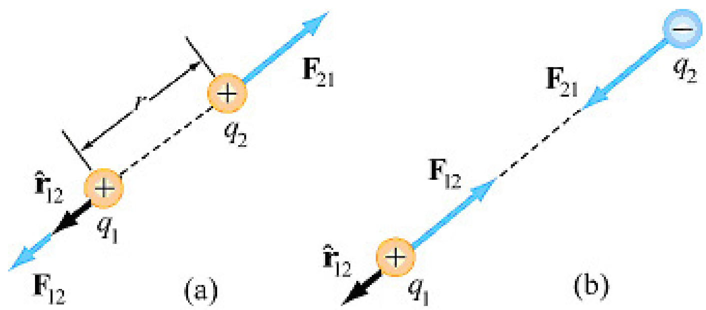


**Figure 17.8** Coulomb interaction between two charges.

The force on the charged particle of charge $q_1$ due to the electric interaction between the two charged particles is given by Coulomb's law,

$$\vec{\mathbf{F}}_{1,2}(\vec{\mathbf{r}}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}\hat{\mathbf{r}}_{1,2}\,. \tag{17.7.2}$$

Coulomb's Law provides an accurate description of the motion of charged particles when they are not bound together. We cannot model the interaction between the electron and the proton when the charged particles are as close together as in the hydrogen atom, since the Newtonian concept of force is not well-defined at length scales associated with the size of atoms. Thus we need a new theory, *quantum mechanics*, to explain the properties of the atom. When the charged particles are far apart they are essentially free particles and the quantum mechanical description of the bound system is not necessary. Therefore the exact same method of solution that was used in the Kepler Problem for orbits of planets applies to the motion of charged particles.

For a proton and an electron in a bound system, the hydrogen atom, Niels Bohr found a semi-classical argument that allows one to use the classical theory of electric forces to predict the observed energies of the hydrogen atom.

The following argument does not satisfy the principles of quantum mechanics, even though the result is in reasonable agreement with experimentally determined properties of the hydrogen atom.

We begin our discussion by recalling our result, Equation (17.4.2) for the energy of the gravitational system of two bodies when considered as a single reduced body of reduced mass $\mu = m_1 m_2 /(m_1 + m_2)$ moving in one dimension, with the distance from the central point denoted by the variable $r$,

$$E = K_{\text{eff}} + U_{\text{eff}} \tag{17.7.3}$$

where the effective potential energy is

$$U_{\text{eff}} = \frac{L^2}{2\mu r^2} + U_{\text{gravity}} = \frac{L^2}{2\mu r^2} - G\frac{m_1 m_2}{r} \tag{17.7.4}$$

and the effective kinetic energy is

$$K_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 . \tag{17.7.5}$$

We can extend this description to the electric interaction between the electron and proton of the hydrogen atom by replacing the constant $-G m_1 m_2$ with $k q_1 q_2$, where the charge of the proton is $q_1 = e$, and the charge of the electron is $q_2 = -e$. The energy is then given by

$$E = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + U_{\text{eff}} = \frac{1}{2}\mu\left(\frac{dr}{dt}\right)^2 + \frac{L^2}{2\mu r^2} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r} . \tag{17.7.6}$$

Since the mass of the electron $m_e = 9.1093826 \times 10^{-31}\ \text{kg}$ is much smaller than the mass of the proton $m_p = 1.67262171 \times 10^{-27}\ \text{kg}$, the reduced mass is approximately the mass of the electron, $\mu \cong m_e$. A schematic plot of the effective potential energy as a function of the variable $r$ for the hydrogen atom is shown in Figure 17.9.

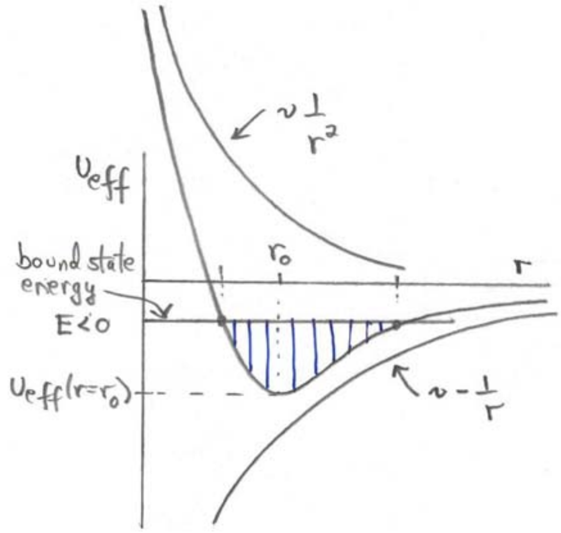


**Figure 17.9** Hydrogen atom energy diagram.

There are two important differences between the classical mechanical description of the possible orbits under a central force and a quantum mechanical description of the possible *states* of a hydrogen atom. The first difference is that the energy $E$ of the hydrogen atom can only take on discrete "quantized" values, unlike the classical case where the energy $E$ can take on a continuous range of values, negative for circular and elliptic "bound" orbits, and positive for hyperbolic "free" orbits. The second difference is that there are additional states with the same value of energy, but which have different "discrete" values of angular momentum, (and other discrete quantum properties of the atom, for example spin of the electron). Chemists identify these discrete values of angular momentum by alphabetical labels $(s, p, d, f, g, ...)$ while physicists label them by the orbital angular momentum quantum numbers $l = 0,1,2,...$.

We shall try to estimate the energy levels of the electron in the hydrogen atom. We shall begin by assuming that the discrete energy states describe circular electron orbits about the proton. (Quantum mechanics requires us to drop the notion that the electron can be thought of as a point particle moving in an orbit and replace the "particle" picture with the idea that the electron's position can only be described by probabilistic arguments.) Despite the unphysical nature of our hypothesis, our estimation of the energy levels of the electron in the atom agree surprisingly well with experiment.

The circular orbits corresponded to the situation described by

$$E_{\min} = \left(U_{\text{eff}}\right)_{\min} = \frac{L^2}{2\mu r^2} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r}. \qquad (17.7.7)$$

This occurs when

$$0 = \frac{dU_{\text{eff}}}{dr} = -\frac{L^2}{\mu r^3} + \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2}\,. \tag{17.7.8}$$

Solving Equation (17.7.8) for the radius $r$ of the orbit, we find

$$r = \frac{4\pi\varepsilon_0 L^2}{\mu e^2}\,. \tag{17.7.9}$$

We also note that the square of the angular momentum is then

$$L^2 = \frac{\mu e^2 r}{4\pi\varepsilon_0}\,. \tag{17.7.10}$$

We now make our semi-classical assumption that the angular momentum $L$ can only assume discrete values

$$L = n\frac{h}{2\pi}\,. \tag{17.7.11}$$

where $n$ is an integer, $n = 1, 2, \ldots$, and $h = 6.6260693\times10^{-34}\ \text{kg}\cdot\text{m}^2\cdot\text{s}^{-1}$ is the *Planck constant*. With this assumption that the values $L$ are discrete, Equation (17.7.10) becomes

$$\left(\frac{n\,h}{2\pi}\right)^2 = \frac{\mu e^2 r_n}{4\pi\varepsilon_0} \tag{17.7.12}$$

where $r_n$ denotes the radii of the discrete circular orbits,

$$r_n = \frac{4\pi\varepsilon_0\, n^2 h^2}{4\pi^2 \mu e^2}\,. \tag{17.7.13}$$

The radii are *quantized*,

$$r_n = \frac{4\pi\varepsilon_0\, n^2 h^2}{4\pi^2 \mu e^2} = n^2 r_1 \tag{17.7.14}$$

and equal to integral multiples of the ground state radius

$$r_1 = \frac{4\pi\varepsilon_0 h^2}{4\pi^2 \mu e^2} \tag{17.7.15}$$

where $r_1$, setting $\mu = m_e$, the electron mass, is

$$r_1 = \frac{\left(6.626\times10^{-34}\,\mathrm{kg\cdot m^2\cdot s^{-1}}\right)^2}{4\pi^2\left(8.988\times10^{9}\,\mathrm{N\cdot m^2\cdot C^{-2}}\right)\left(9.109\times10^{-31}\,\mathrm{kg}\right)\left(1.602\times10^{-19}\mathrm{C}\right)^2} \quad (17.7.16)$$
$$= 5.292\times10^{-11}\,\mathrm{m}$$

to four significant figures. The length $r_1$ is known as the "Bohr radius" and is often given as $a_0$ or, perhaps counterintuitively, $a_\infty$, the latter notation indicating a nuclear mass of infinity, in which case, $\mu = m_e$, as in the above calculation.

We can substitute Equation (17.7.16) into Equation (17.7.7) and find that the energy levels are also quantized and given by

$$\begin{aligned} E_n &= \frac{L^2}{2\mu r_n^2} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{r_n} = \frac{1}{4\pi\varepsilon_0}\left(\frac{\mu e^2 r_n}{2\mu r_n^2} - \frac{e^2}{r_n}\right) \\ &= -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r_n} = -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{2r_1}\frac{1}{n^2}. \end{aligned} \qquad (17.7.17)$$

Using Equation (17.7.15) for $r_1$, Equation (17.7.17) for the energy levels becomes

$$E_n = -\frac{1}{4\pi\varepsilon_0}\frac{1}{2}\frac{e^2}{4\pi\varepsilon_0 n^2 h^2 / 4\pi^2 \mu e^2} = -\frac{2\pi^2 \mu e^4}{\left(4\pi\varepsilon_0\right)^2 h^2}\frac{1}{n^2} = -\frac{A}{n^2} \qquad (17.7.18)$$

where, with $\mu = m_e$, the constant $A$ is given by

$$\begin{aligned} A &= \frac{2\pi^2 m_e e^4}{\left(4\pi\varepsilon_0\right)^2 h^2} \\ &= \frac{2\pi^2\left(9.109\times10^{-31}\,\mathrm{kg}\right)\left(8.988\times10^{9}\,\mathrm{N\cdot m^2\cdot C^{-2}}\right)^2\left(1.602\times10^{-19}\mathrm{C}\right)^4}{\left(6.626\times10^{-34}\,\mathrm{kg\cdot m^2\cdot s^{-1}}\right)^2} \quad (17.7.19) \\ &= 2.180\times10^{-18}\,\mathrm{J} \end{aligned}$$

to four significant figures. We can express this energy in terms of the energy units of electron-volts ($\mathrm{eV}$), where an electron-volt is the energy necessary to accelerate an electron with charge $e$ across a potential energy per unit charge of 1 volt ( $1$ volt $=1$ joule per coulomb). Thus $1\,\mathrm{eV} = \left(1.602\times10^{-19}\,\mathrm{C}\right)\left(1\,\mathrm{J\cdot C^{-1}}\right) = 1.602\times10^{-19}\,\mathrm{J}$ and

$$A = \left(2.180\times10^{-18}\text{ J}\right)\frac{1\text{eV}}{\left(1.602\times10^{-19}\text{ J}\right)} = 1.361\times10^{1}\text{ eV}. \qquad (17.7.20)$$

The energy in Equation (17.7.18) can be written as

$$E_n = -\frac{hc\,R}{n^2} \qquad (17.7.21)$$

where $R$, the *Rydberg constant*, is given by

$$R = \frac{A}{hc} = \frac{\left(2.180\times10^{-18}\text{ J}\right)}{\left(6.626\times10^{-34}\text{ kg}\cdot\text{m}^2\cdot\text{s}^{-1}\right)\left(2.998\times10^{8}\text{m}\cdot\text{s}^{-1}\right)} \qquad (17.7.22)$$
$$= 1.097\times10^{-7}\,\text{m}^{-1}.$$

and $c = 2.998\times10^{8}\text{ m}\cdot\text{s}^{-1}$ is the speed of light.

The first few energy levels are shown in Figure 17.10, along with the energy difference between the second and third levels. The first three energy levels are $E_1 = -13.6\text{ eV}$, $E_2 = -3.39\text{ eV}$, $E_3 = -1.50\text{ eV}$.

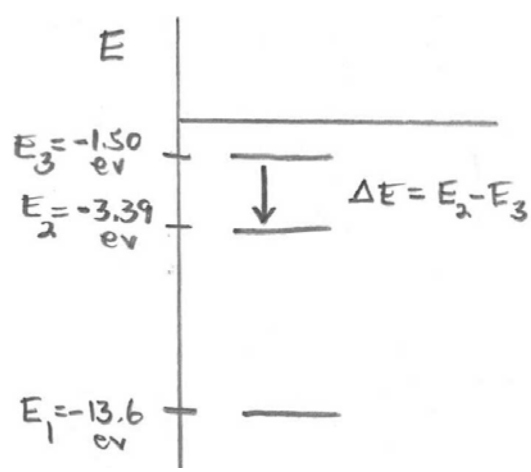


**Figure** 17.10 Energy levels for an electron in a hydrogen atom

**Emission of light**

When an electron makes a transition from a higher energy state $E_i$ to a lower energy state $E_f$, light is emitted. The frequency of the emitted light is given by

$$f = -\frac{\Delta E}{h} = -\frac{E_f - E_i}{h}. \qquad (17.7.23)$$

Using our result as given in Equation (17.7.18) for the energy levels, we have

$$f = Rc\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right). \qquad (17.7.24)$$

The wavelength of the emitted light $\lambda$ is related to the frequency $f$ of the light by

$$f\lambda = c\,. \qquad (17.7.25)$$

Thus the inverse wavelength of the light is given by

$$\frac{1}{\lambda} = \frac{f}{c} = R\left(\frac{1}{n_f^2} - \frac{1}{n_1^2}\right). \qquad (17.7.26)$$

**17.7.1 Example:** Calculate the wavelength of the light emitted when an electron in the energy level $n = 3$ drops to the energy level $n = 2$.

From Equation (17.7.26) the wavelength is

$$\lambda_{3,2} = \frac{1}{R}\left(\frac{n_i^2 n_f^2}{n_i^2 - n_f^2}\right) = 6.561\times 10^{-7}\ \text{m}\,. \qquad (17.7.27)$$

The emitted light lies in the visible spectrum and appears red to the human eye.