Circular Motion
Kinematics
Announcements

Open up the Daily Concept Questions page on the MITx 8.01x Webpage.

Sunday Tutoring in 26-152 from 1-5 pm

Week 3 Day 1 Prepset 1 due Mon at 8:30 am

Problem Set 3 due Tue Week 4 at 9 pm

Week 3 Day 2 Prepset 2 due Wed at 8:30 am

Exam 1 Thursday Sept 26 7:30-9:30 Rooms see Exam 1 8.01x webpage
W03D1 Learning Outcomes

• Understand how to construct a polar coordinate system with unit vectors and express those unit vectors in terms of Cartesian unit vectors (conceptual and methodological)

• Understand why polar coordinates are the best coordinates for two dimension central motion (conceptual)

• Understand how to express position, velocity, and acceleration in polar coordinates (conceptual and methodological)

• Understand the vector nature of angular velocity (conceptual)

• Know how to take a derivative of a vector that is constant in magnitude but changes direction (methodological)

• Understand the meaning of centripetal acceleration and tangential acceleration (conceptual)
Examples of Circular Motion

- Ball on a string of fixed length rotating in a circle
- Any point on a rotating wheel about a fixed axis
- Any point on the rotating Earth
- Satellite rotating around Earth (generally circular)
- Moon rotating around Earth (nearly circular)
- Nearly circular planetary orbits about center of mass of sun (center of mass reference) (nearly circular)
- Sun rotating about center of mass of galaxy (center of mass reference)
Polar Coordinate System

**Coordinates** $$(r, \theta)$$

**Unit vectors** $$(\hat{\mathbf{r}}, \hat{\theta})$$

**Relation to Cartesian Coordinates**

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}(y / x)$$
Measure of Angle: Radian

Angles are dimensionless and measured in radians.

When $s = r$, the angle $\theta$ is defined to be one radian. Therefore $2\pi$ radians $= 360^\circ$.

When $r = 1$ (unit circle) then the angle $\theta$ measured in radians is equal to the arclength $s$, $\theta = s$. 
Group Problem: Unit Vector Coordinate Transformations

Find an expression for the unit vectors in polar coordinates in terms of the Cartesian unit vectors shown in the figure.
Unit Vector Coordinate Transformations

Transformations between unit vectors in polar coordinates and Cartesian unit vectors

\[
\hat{r}(t) = \cos \theta(t) \hat{i} + \sin \theta(t) \hat{j}
\]
\[
\hat{\theta}(t) = -\sin \theta(t) \hat{i} + \cos \theta(t) \hat{j}
\]
\[
\hat{i} = \cos \theta(t) \hat{r}(t) - \sin \theta(t) \hat{\theta}(t)
\]
\[
\hat{j} = \sin \theta(t) \hat{r}(t) + \cos \theta(t) \hat{\theta}(t)
\]
Circular Motion: Position

Position Vector

\[ \vec{r}(t) = r \hat{r}(t) = r (\cos \theta(t) \hat{i} + \sin \theta(t) \hat{j}) \]
CQ Time Derivative of Position Vector for Circular Motion

A point-like object undergoes circular motion at a constant speed. The vector from the center of the circle to the object

1. has constant magnitude and hence is constant in time.

2. has constant magnitude but is changing direction so is not constant in time.

3. is changing in magnitude and hence is not constant in time.
Circular Motion: Velocity

Magnitude of the Displacement

\[ |\Delta \vec{r}| = r |\Delta \theta| \]

Magnitude of Velocity

\[ |\vec{v}(t)| = \lim_{\Delta t \to 0} \frac{|\Delta \vec{r}|}{\Delta t} = \lim_{\Delta t \to 0} \frac{r |\Delta \theta|}{\Delta t} = r \left| \frac{d\theta}{dt} \right| \]

Direction of the velocity:
The tangent line at the point \( P \) defines a direction in space, up to a sign. The velocity points along this direction, again up to a sign that depends on the direction of the rotation of the object.
Chain Rule of Differentiation

Recall that when taking derivatives of a differentiable function $f = f(\theta)$ whose argument is also a differentiable function $\theta = g(t)$ then $f = f(g(t)) = h(t)$ is a differentiable function of $t$ and

$$\frac{df}{dt} = \frac{df}{d\theta} \frac{d\theta}{dt}$$

Examples:

$$\frac{d}{dt} \cos \theta(t) = -\sin \theta \frac{d\theta}{dt}$$

$$\frac{d}{dt} \sin \theta(t) = \cos \theta \frac{d\theta}{dt}$$
Group Problem: Algebraic Derivation of Velocity for Circular Motion

Derive velocity in polar coordinates.

1. Begin by expressing position vector in Cartesian coordinates

\[ \mathbf{r}(t) = r \hat{r}(t) \]

2. Differentiate using chain rule

\[ \frac{df}{dt} = \frac{df}{d\theta} \frac{d\theta}{dt} \]

3. Then use unit vector coordinate transformations to express result in polar coordinates

\[ \mathbf{r}(t) = \cos \theta(t) \hat{i} + \sin \theta(t) \hat{j} \]

\[ \dot{\theta}(t) = -\sin \theta(t) \hat{i} + \cos \theta(t) \hat{j} \]
Group Problem: Velocity for Circular Motion Solution

Velocity: \[ \vec{v}(t) = r \frac{d\theta}{dt} \hat{\theta}(t) \]

Calculation:

\[
\vec{v}(t) = \frac{d \vec{r}(t)}{dt} = \frac{d}{dt} (r \hat{\theta}(t)) \\
= r \frac{d}{dt} \left( \cos \theta(t) \hat{i} + \sin \theta(t) \hat{j} \right) \\
= r \left( -\sin \theta(t) \frac{d\theta}{dt} \hat{i} + \cos \theta(t) \frac{d\theta}{dt} \hat{j} \right) = r \frac{d\theta}{dt} \hat{\theta}(t)
\]
Summary: Derivatives of Unit Vectors in Polar Coordinates

\[ \hat{r}(t) = \cos \theta(t) \hat{i} + \sin \theta(t) \hat{j} \]

\[ \frac{d}{dt} \hat{r}(t) = \frac{d\theta}{dt} \hat{\theta}(t) \]

\[ \hat{\theta}(t) = -\sin \theta(t) \hat{i} + \cos \theta(t) \hat{j} \]

\[ \frac{d}{dt} \hat{\theta}(t) = \begin{pmatrix} -\cos \theta(t) \frac{d\theta}{dt} \hat{i} - \sin \theta(t) \frac{d\theta}{dt} \hat{j} \end{pmatrix} \]

\[ \frac{d}{dt} \hat{\theta}(t) = -\frac{d\theta}{dt} \hat{r}(t) \]
Derivative of a Vector of Constant Length but Changing Direction

Vector

\[ \vec{A}(t) = |\vec{A}| \hat{A} \]

Time Derivative:

\[ \frac{d\vec{A}(t)}{dt} = |\vec{A}| \frac{d\theta}{dt} \hat{A}_\perp \]

\[ \frac{d\vec{A}}{dt} = |\vec{A}| \frac{d\theta}{dt} \hat{A}_\perp \]
If the positive z-axis points up, then we choose $\theta$ to be increasing in the counterclockwise direction
Fixed Axis Rotation: Angular Velocity

Angle variable
SI unit: \( \theta \) [rad]

Angular velocity \( \vec{\omega} \equiv \omega_z \hat{k} \equiv (d\theta / dt) \hat{k} \)
SI unit: \[ \text{rad} \cdot \text{s}^{-1} \]

Component: \( \omega_z \equiv d\theta / dt \)

Magnitude \( \omega \equiv |\omega_z| \equiv |d\theta / dt| \)
Direction of Angular Velocity

\[ \ddot{\omega} \equiv \omega_z \hat{k} \equiv (d \theta / dt) \hat{k} \]

\[ \omega_z \equiv d \theta / dt \]

\( \omega_z \equiv d \theta / dt > 0 \), direction +\( \hat{k} \)

\( \omega_z \equiv d \theta / dt < 0 \), direction −\( \hat{k} \)
Demo: Bicycle Wheel

Demonstrate angular velocity and angular acceleration
Speed and Angular Speed

The tangential component of the velocity of the object undergoing circular motion is proportional to the rate of change of the angle with time:

\[ v_\theta = r \frac{d\theta}{dt} \equiv r\omega \]

Angular speed:

\[ \omega \equiv \left \lvert \frac{d\theta}{dt} \right \rvert \quad \text{(units: rad \cdot s}^{-1}) \]

Velocity:

\[ \vec{v}(t) = v_\theta \hat{\theta} = r\omega_z \hat{\theta} \quad \vec{v} = \vec{\omega} \times \vec{r} = \frac{d\theta}{dt} \hat{k} \times r\hat{r} = r \frac{d\theta}{dt} \hat{\theta} \]

Speed:

\[ v = r\omega \]
Object A sits at the outer edge (rim) of a merry-go-round, and object B sits halfway between the rim and the axis of rotation. The merry-go-round makes a complete revolution once every thirty seconds. The magnitude of the angular velocity of Object B is

1. half the magnitude of the angular velocity of Object A.

2. the same as the magnitude of the angular velocity of Object A.

3. twice the the magnitude of the angular velocity of Object A.
A wooden ball is attached to the rim of a spinning wheel. The ball is held in place by a string. When the string is cut, the ball flies in a straight tangent to the wheel in the vertical direction.

A wheel is connected via a pulley to a motor. A thread is knotted and placed through a hole in a ping pong ball of mass $m$. A wing nut secures the thread holding the ball a distance $R$ from the center of the wheel. The wheel is set in motion. When a satisfactory angular speed $\omega$ is reached, the string is cut and the ball comes off at a tangent to the spinning wheel, traveling vertically upward a distance $h$. Find the angular speed of the rotating wheel just before the string is cut.
Circular Motion: Constant Speed, Period, and Frequency

In one period the object travels a distance equal to the circumference:

\[ s = 2\pi R = vT \]

Period: the amount of time to complete one circular orbit of radius \( R \)

\[ T = \frac{2\pi R}{v} = \frac{2\pi R}{R\omega} = \frac{2\pi}{\omega} \]

Frequency is the inverse of the period:

\[ f = \frac{1}{T} = \frac{\omega}{2\pi} \] (units: \( s^{-1} \) or Hz)
Group Problem: Angular Velocity

A particle is moving in a circle of radius $R$. At $t = 0$, it is located on the $x$-axis. The angle the particle makes with the positive $x$-axis is given by

$$\theta(t) = At - Bt^3$$

where $A$ and $B$ are positive constants. Determine the (a) angular velocity vector, and (b) the velocity vector (express your answers in cylindrical coordinates). (c) At what time $t = t_1$ is the angular velocity zero? (d) What is the direction of the angular velocity for (i) $t < t_1$, and (ii) $t > t_1$?
Acceleration and Circular Motion

When an object moves in a circular orbit, the direction of the velocity changes and the speed may change as well.

\[
\vec{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \equiv \frac{d\vec{v}}{dt}
\]

For circular motion, the acceleration will always have a non-positive radial component \((a_r)\) due to the change in direction of velocity, (it may be zero at the instant the velocity is zero).

The acceleration may have a tangential component if the speed changes \((a_t)\). When \(a_t = 0\), the speed of the object remains constant.
Circular Motion: Acceleration

Definition of acceleration

\[ \vec{a}(t) \equiv \frac{d\vec{v}(t)}{dt} = \frac{d}{dt} \left( r \frac{d\theta}{dt}(t) \hat{\theta}(t) \right) \]

Use product rule

\[ \vec{a}(t) = r \frac{d^2\theta}{dt^2} \hat{\theta}(t) + r \frac{d\theta}{dt}(t) \frac{d\hat{\theta}(t)}{dt} \]

Use differentiation rule

\[ \frac{d\hat{\theta}(t)}{dt} = \frac{d\theta}{dt}(-\hat{r}(t)) \]

Result

\[ \vec{a}(t) = r \frac{d^2\theta}{dt^2} \hat{\theta}(t) + r \left( \frac{d\theta}{dt} \right)^2 (-\hat{r}(t)) \equiv a_\theta \hat{\theta}(t) + a_r \hat{r}(t) \]
CQ Circular Motion

As the object speeds up along the circular path in a counterclockwise direction shown below, its acceleration points in direction

1. toward the center of the circular path.
2. in a direction tangential to the circular path.
3. in a direction towards the upper left as shown in figure.
4. in a direction towards the lower left as shown in figure.
Fixed Axis Rotation: Angular Acceleration

Angular acceleration

\[ \vec{\alpha} \equiv \alpha_z \hat{k} \equiv \left( \frac{d^2 \theta}{dt^2} \right) \hat{k} \]

SI unit: \[ \left[ \text{rad} \cdot \text{s}^{-2} \right] \]

Component: \( \alpha_z \equiv \frac{d^2 \theta}{dt^2} = \frac{d\omega_z}{dt} \)

Magnitude \( \alpha = |\alpha_z| = \left| \frac{d^2 \theta}{dt^2} \right| \)

Direction

\[ \alpha_z \equiv \frac{d^2 \theta}{dt^2} > 0, \text{ direction } +\hat{k} \]
\[ \alpha_z \equiv \frac{d^2 \theta}{dt^2} < 0, \text{ direction } -\hat{k} \]
Direction of Angular Acceleration

\[ \vec{\alpha} \equiv \alpha_z \hat{k} \equiv \left( \frac{d^2 \theta}{dt^2} \right) \hat{k} \]

\[ \alpha_z \equiv \frac{d^2 \theta}{dt^2} > 0, \text{ direction } +\hat{k} \]

\[ \alpha_z \equiv \frac{d^2 \theta}{dt^2} < 0, \text{ direction } -\hat{k} \]
Constant Speed Circular Motion: Centripetal Acceleration

Position
\[ \vec{r}(t) = r \hat{r}(t) = r(\cos \theta(t) \hat{i} + \sin \theta(t) \hat{j}) \]

Angular Speed
\[ \omega \equiv \left| \frac{d\theta}{dt} \right| = \text{constant} \]

Velocity
\[ \vec{v}(t) = v_\theta \hat{\theta}(t) = r(\frac{d\theta}{dt})\hat{\theta}(t) \]

Speed
\[ v = \left| \vec{v}(t) \right| = r \left| \frac{d\theta}{dt} \right| = r\omega \]

Zero Angular Acceleration
\[ \alpha_z \equiv \frac{d^2\theta}{dt^2} = \frac{d\omega_z}{dt} = 0 \]

Radial Acceleration
\[ \vec{a}(t) = a_r \hat{r}(t) = -r \left( \frac{d\theta}{dt} \right)^2 \hat{r}(t) = -r\omega^2 \hat{r}(t) = -(\frac{v^2}{r})\hat{r}(t) \]
Alternative Forms of Magnitude of Centripetal Acceleration

Parameters: speed $v$, angular speed $\omega$, frequency $f$, period $T$

$$|a_r| = \frac{v^2}{r} = r\omega^2 = r(2\pi f)^2 = \frac{4\pi^2 r}{T^2}$$
Circular Motion: Tangential Acceleration

When the component of the angular velocity is a function of time,

\[ \omega_z(t) = \frac{d\theta}{dt}(t) \]

The component of the velocity has a non-zero derivative

\[ \frac{dv_\theta(t)}{dt} = r \frac{d^2\theta}{dt^2}(t) \]

Then the *tangential acceleration* is the time rate of change of the magnitude of the velocity

\[ \bar{a}_\theta(t) = a_\theta(t)\dot{\theta}(t) = r \frac{d^2\theta}{dt^2}(t)\dot{\theta}(t) = r\alpha_z(t)\dot{\theta}(t) \]
CQ Circular Motion

An object moves counter-clockwise along the circular path shown below. As it moves along the path its acceleration vector continuously points toward point S. The object

1. speeds up at $P$, $Q$, and $R$.
2. slows down at $P$, $Q$, and $R$.
3. speeds up at $P$ and slows down at $R$.
4. slows down at $P$ and speeds up at $R$.
5. speeds up at $Q$.
6. slows down at $Q$.
7. No object can execute such a motion.
Group Problem: Circular Motion

A particle is moving in a circle of radius \( R \). At \( t = 0 \), it is located on the \( x \)-axis. The angle the particle makes with the positive \( x \)-axis is given by

\[
\theta(t) = At^3 - Bt
\]

where \( A \) and \( B \) are positive constants. Determine the (a) velocity vector, and (b) acceleration vector (express your answers in polar coordinates). (c) At what time is the centripetal acceleration zero?
Summary: Circular Motion: Vector Description

Position

\[ \mathbf{r}(t) = r \hat{r}(t) \]

Component of Angular Velocity

\[ \omega_z \equiv \frac{d\theta}{dt} \]

Velocity

\[ \mathbf{v} = v_\theta \hat{\theta}(t) = r \left( \frac{d\theta}{dt} \right) \hat{\theta} \]

Component of Angular Acceleration

\[ \alpha_z \equiv \frac{d\omega_z}{dt} = \frac{d^2\theta}{dt^2} \]

Acceleration

\[ \mathbf{a} = a_r \hat{r} + a_\theta \hat{\theta} \]

\[ a_r = -r \left( \frac{d\theta}{dt} \right)^2 = -\left( \frac{v^2}{r} \right), \quad a_\theta = r \left( \frac{d^2\theta}{dt^2} \right) \]
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• Understand why polar coordinates are the best coordinates for two dimension central motion (conceptual)

• Understand how to express position, velocity, and acceleration in polar coordinates (conceptual and methodological)

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• Know how to take a derivative of a vector that is constant in magnitude but changes direction (methodological)

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