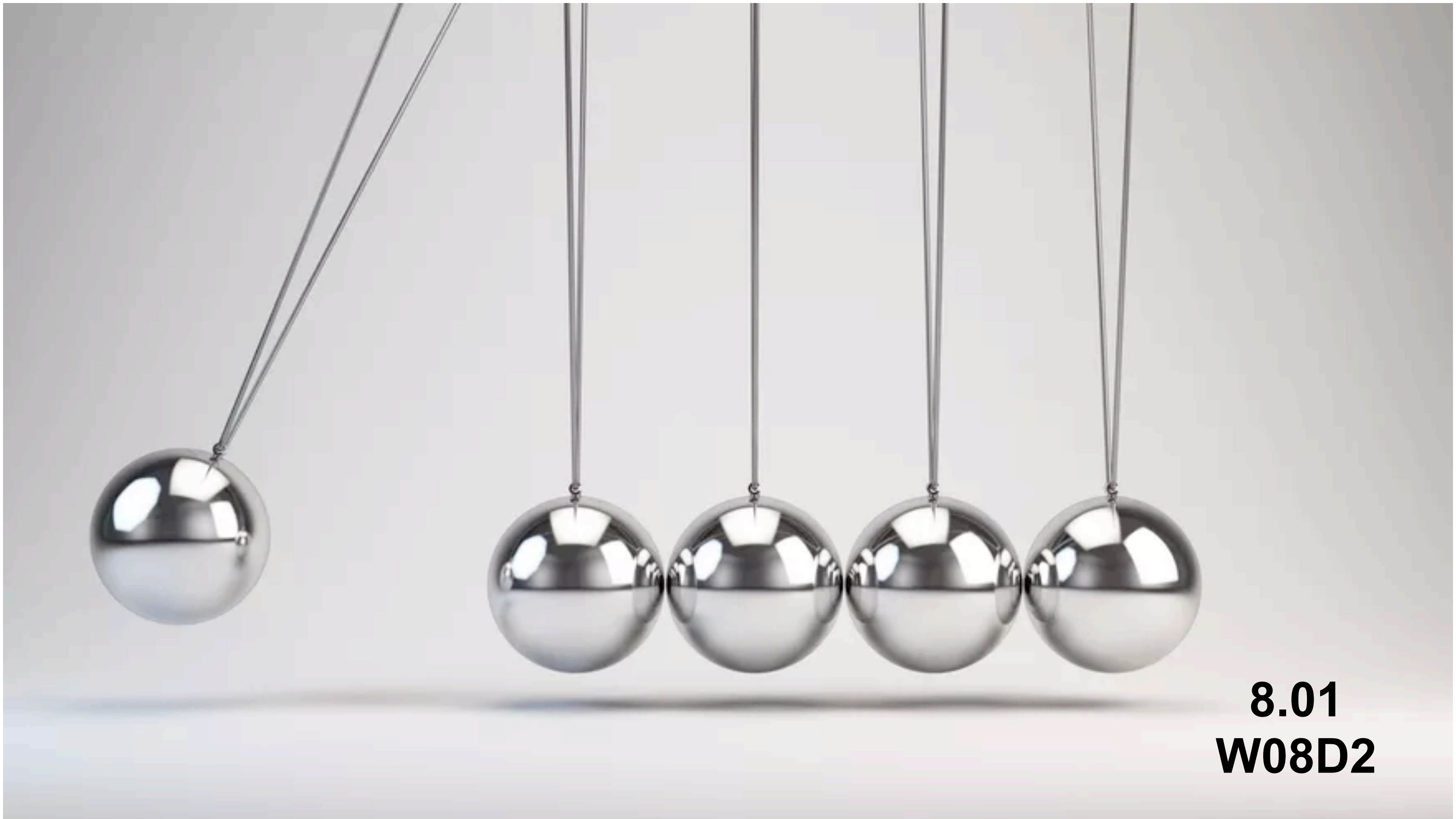
Simple Harmonic Motion and and Collision Theory



Announcements

Week 8: Problem Set 7 due Wednesday at 11 pm

Week 8: There is Friday Problem Solving

Week 9:
LS 1 and LS 2 due Sunday at 11 pm
LS 3 and LS 4 due Tuesday at 11 pm

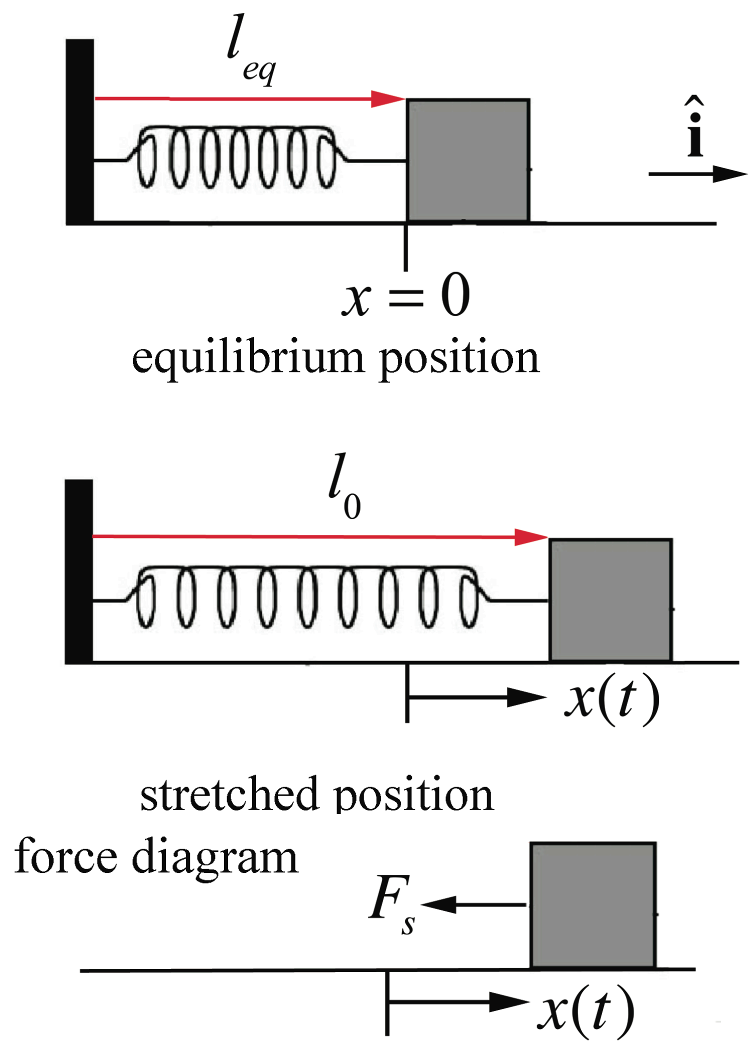
Week 9: Problem Set 8 due Wednesday at 11 pm

Simple Harmonic Motion

Demo Mass on a Spring: Simple Harmonic Motion Energy

Hooke's Law and the Simple Harmonic Oscillator Equation (SHO):

Define system, choose coordinate system.



Draw free-body diagram.

Hooke' s Law: $\vec{\mathbf{F}}_{\text{spring}} = -kx\,\hat{\mathbf{i}}$

Apply Newton's Second Law:

$$-kx = m\frac{d^2x}{dt^2}$$

Simple Harmonic Oscillator Equation

$$\vec{\mathbf{F}}_s = -F_s\hat{\mathbf{i}} = -k\,x\,\hat{\mathbf{i}}$$

Concept Q.: Simple Harmonic Motion

Which of the functions *x*(*t*) below has a second derivative which is proportional to the negative of the function *x*(*t*)

$$\frac{d^2x}{dt^2} \propto -x?$$

Note *t* is a real number.

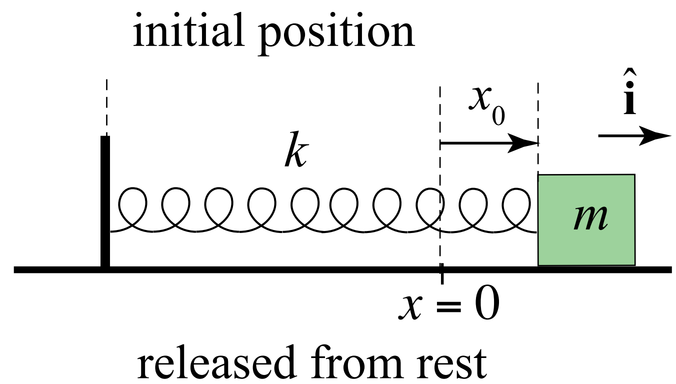
1. $x(t) = \frac{1}{2}at^2$
2. $x(t) = Ae^{t/T}$

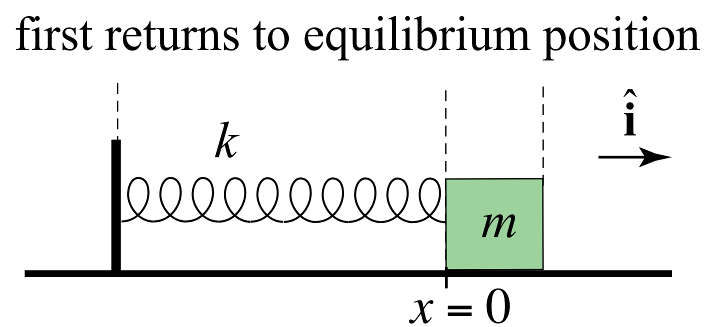
.

3. $x(t) = Ae^{-t/T}$
4. $x(t) = A\cos\left(\frac{2\pi}{T}t\right)$

CQ SMO: Velocity

A block of mass m, attached to a spring with spring constant k, is free to slide along a horizontal frictionless surface. At t = 0 the block-spring system is stretched an amount $x_0 > 0$ from the equilibrium position and is released from rest. Let T be the period of oscillation.





What is the x -component of the velocity of the block when it first comes back to the equilibrium position?

1. $v_x = -x_0 \dfrac{4}{T}$
2. $v_x = +x_0 \dfrac{4}{T}$
3. $v_x = -\sqrt{\dfrac{k}{m}} x_0$
4. $v_x = +\sqrt{\dfrac{k}{m}} x_0$

Group Problem: Angular Frequency

For a spring of spring constant *k* and a block of mass *m*, **determine the constan** **ω_0, called the angular frequency,** such that

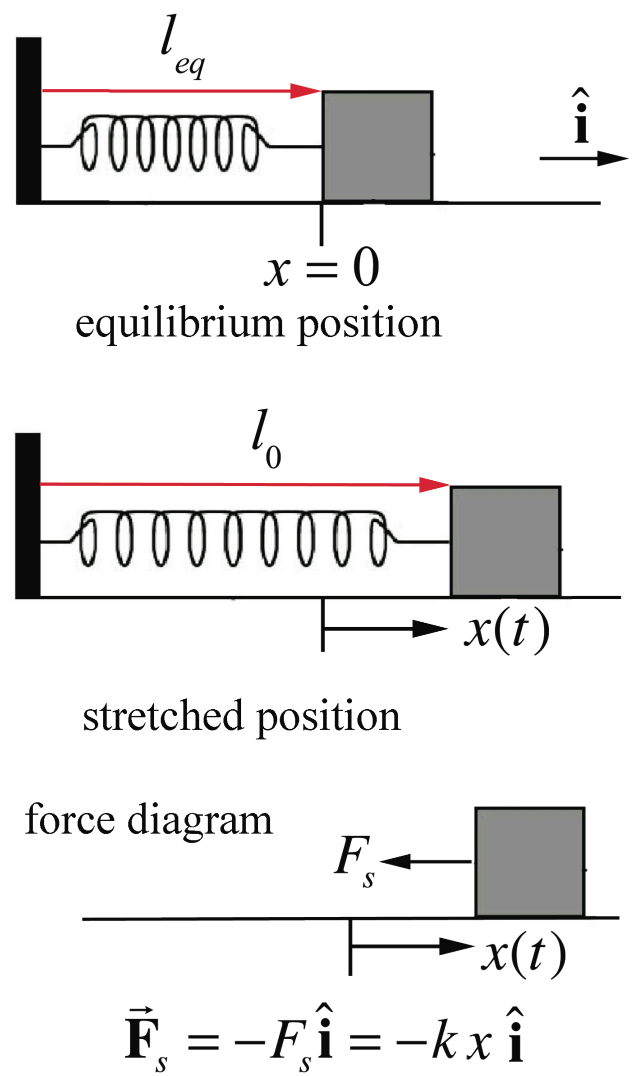
$$x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

where C and D are constants.

is a solution to the differential equation

$$-kx = m\frac{d^2x}{dt^2}.$$

Hint: Calculate the first and second derivatives, then substitute 2nd derivative into differential equation and solve for ω_0 . Remember to use the chain rule.



Initial Conditions: SHO

Equation of Motion: $$-kx = m\frac{d^2x}{dt^2}$$

Period [s]: $$T = 2\pi / \omega_0 = 2\pi\sqrt{m/k}$$

Angular Frequency [rad/s]: $$\omega_0 = \sqrt{k/m}$$

Frequency [Hz]: $$f = 1/T = \omega_0 / 2\pi = \sqrt{k/m} / 2\pi$$

Position: $$x(t) = C\cos(\omega_0 t) + D\sin(\omega_0 t)$$

Velocity: $$v_x(t) = dx/dt = -\omega_0 C\sin(\omega_0 t) + \omega_0 D\cos(\omega_0 t)$$

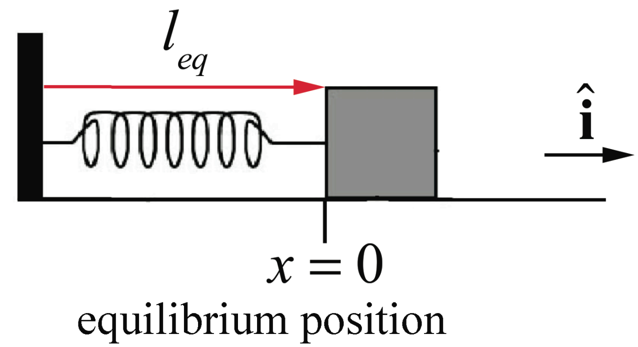
One of your W08D3 FPS problems will guide you through the calculations to show that

$$x_0 \equiv x(t=0) = C$$

$$v_{x,0} \equiv v_x(t=0) = \omega_0 D$$

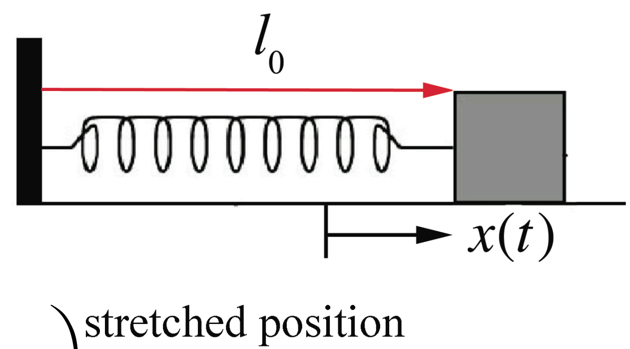
Energy and Simple Harmonic Motion

A block of mass is attached to spring with spring constant *k* . The block slides on a frictionless surface. When the block is at position *x*(*t*) moving with speed $v_x(t)$, the energy is constant and so its derivative is zero



$$E = K(t) + U(t)$$

$$E = (1/2)m(v_x(t))^2 + (1/2)k(x(t))^2$$



Set dE/dt = 0 and use chain rule:

$$0 = dE/dt = mv_x \frac{dv_x}{dt} + kx\frac{dx}{dt} = v_x\left(m\frac{d^2x}{dt^2} + kx\right)$$

Two Solutions: trivial solution $v_x = 0$;

Simple harmonic oscillator eq.: $0 = m\dfrac{d^2x}{dt^2} + kx$

Generalization: SHO and Potential Energy

Consider a system in which the **mechanical energy is constant**. If the potential energy of the system is quadratic in a position variable

$$U = (1/2)bx^2$$

where b is a positive constant then the energy

$$E = (1/2)mv^2 + (1/2)bx^2$$

The derivative with respect to time is zero

$$0 = \frac{dE}{dt} = mv\frac{dv}{dt} + bx\frac{dx}{dt} = v\left(m\frac{d^2x}{dt^2} + bx \right).$$

Therefore the non-trivial (i.e. $v \neq 0$) case is the SHO equation

$$d^2x / dt^2 = -(b/m)x.$$

Hence the angular frequency by comparison to the spring equation is

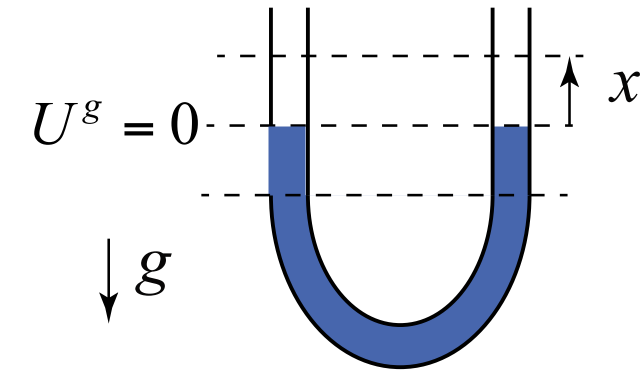
$$\omega_0 = \sqrt{b/m}$$

Demonstration:
U-tube Oscillations C15

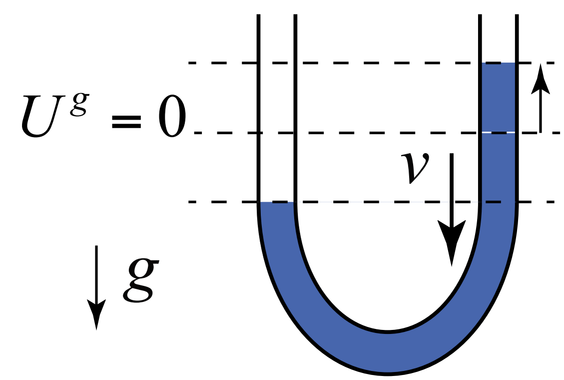
Ex.: Fluid Oscillations in a U-tube

A U-tube open at both ends to atmospheric pressure is filled with an incompressible fluid of density r. The cross-sectional area A of the tube is uniform and the total length of the column of fluid is L. At time t, one side is raised a height x above the equilibrium position, and all the mass is flowing with speed v.

What is the frequency of the ensuing simple harmonic motion? Assume streamline flow and no drag at the walls of the U-tube. The gravitational constant is g.



equilibrium configuration

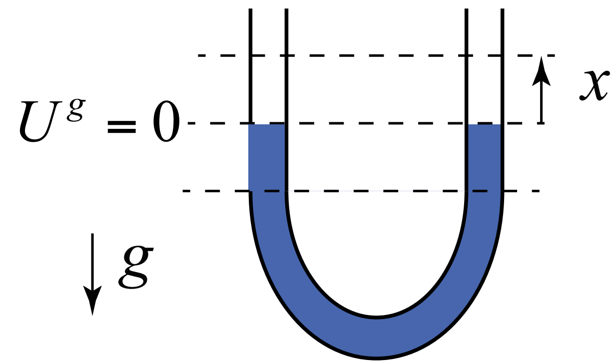


configuration at time t

WE Sol.: Fluid Oscillations in a U-tube

Set $U^g = 0$ at the equilibrium configuration.
Then at time t

$$U^g = \Delta m g x$$

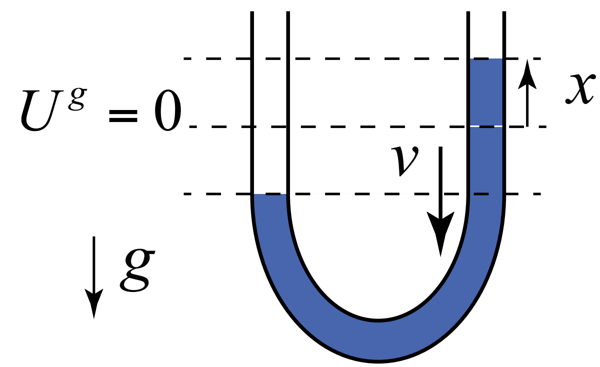


equilibrium configuration

Raised mass:

$$\Delta m = \rho A x \Rightarrow$$

$$U^g = \rho A g x^2 = (1/2) 2 \rho A g x^2 = (1/2) b x^2$$



configuration at time t

Kinetic energy of all the moving water:

$$K = (1/2) m v^2 = (1/2) \rho A L v^2$$

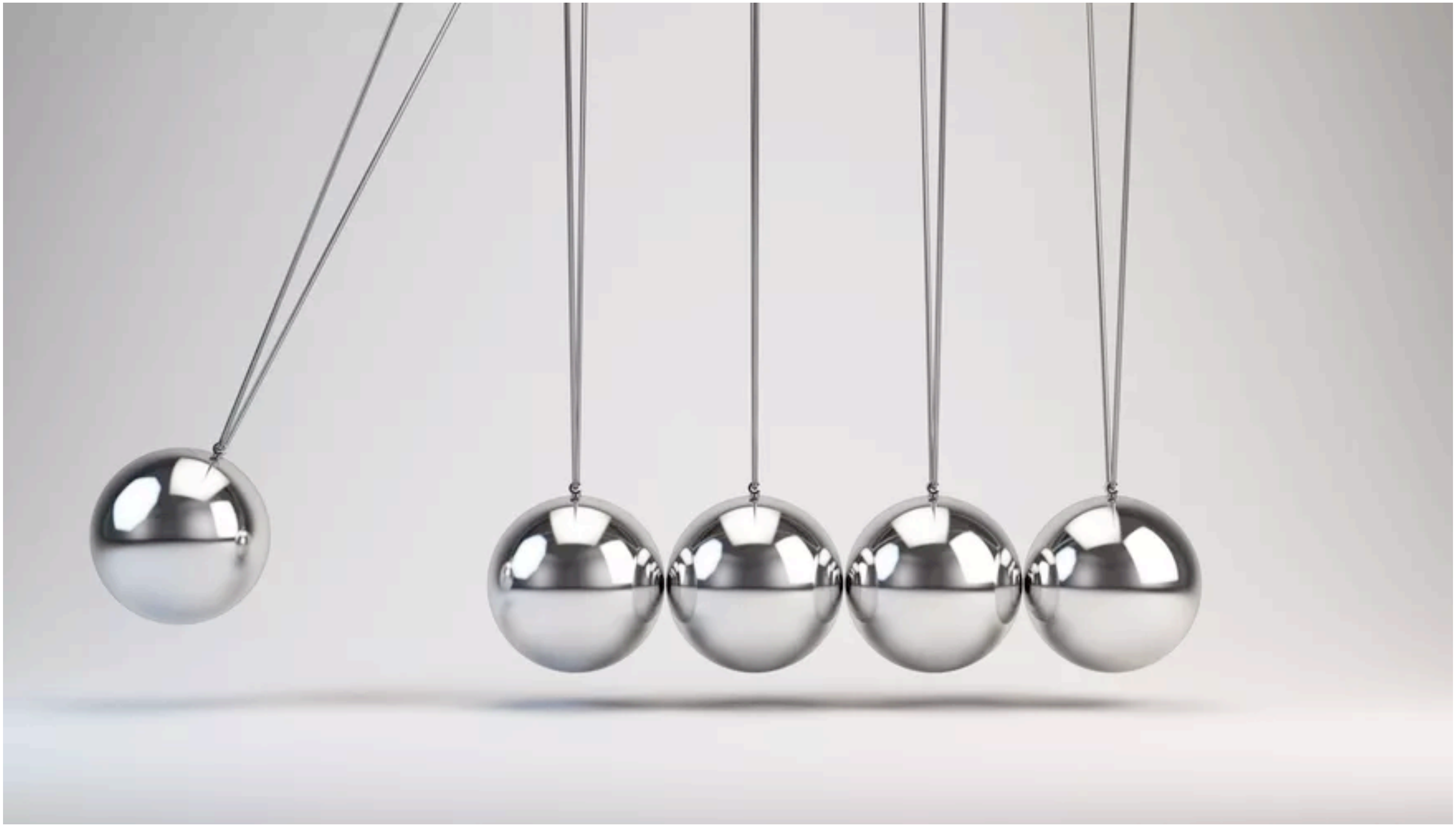
Energy:

$$E = K + U = (1/2) m v^2 + (1/2) b x^2$$

Angular frequency:

$$\omega_0 = \sqrt{\frac{b}{m}} = \sqrt{\frac{2\rho A g}{\rho A L}} = \sqrt{\frac{2g}{L}}$$

Demo Collisions



Collisions

Any interaction between (usually two) objects which occurs for short time intervals when interaction forces dominate over external forces.

Examples:

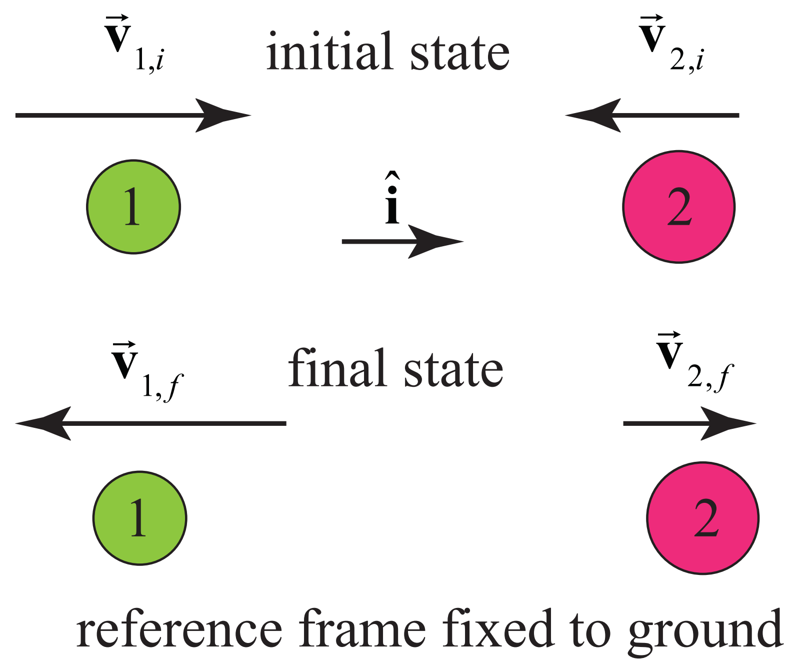
Collisions of motor vehicles.

Collisions of subatomic particles – collisions allow study force law.

Collisions in sports: medical injuries, projectiles, etc.

Collision 1-Dim.: Momentum Principle

Two particles interact with no external forces along direction of motion:



Momentum principle in components:

$$p_{x,i}^{sys} = p_{x,f}^{sys}$$

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f}$$

$$m_1 (v_{1,x,i} - v_{1,x,f}) = m_2 (v_{2,x,f} - v_{2,x,i})$$

Collision Theory: Energy

Types of Collisions

Elastic:

$$K_i^{\text{sys}} = K_f^{\text{sys}}$$

$$\frac{1}{2}m_1 v_{1,i}{}^2 + \frac{1}{2}m_2 v_{2,i}{}^2 + \cdots = \frac{1}{2}m_1 v_{1,f}{}^2 + \frac{1}{2}m_2 v_{2,f}{}^2 + \cdots$$

Inelastic:

$$K_i^{\text{sys}} > K_f^{\text{sys}}$$

Completely Inelastic: Only one body emerges.

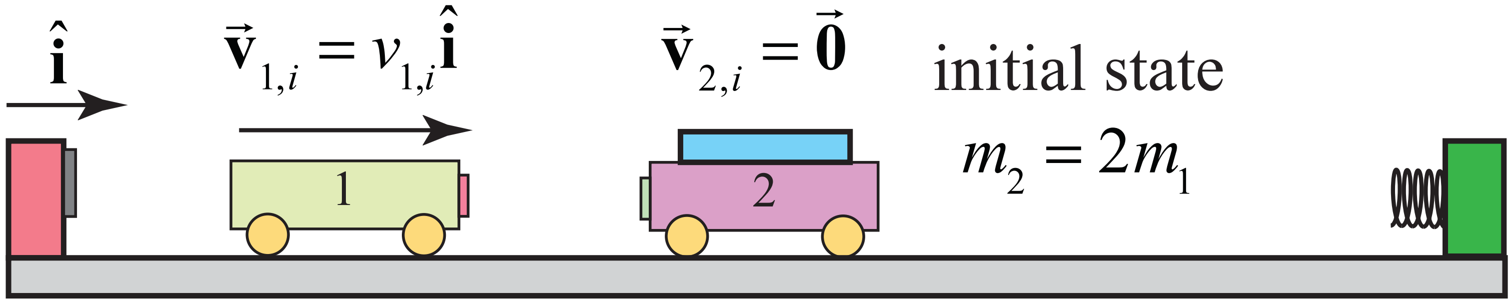
Superelastic:

$$K_i^{\text{sys}} < K_f^{\text{sys}}$$

Demo:
Elastic and Inelastic Collisions

Carts on track

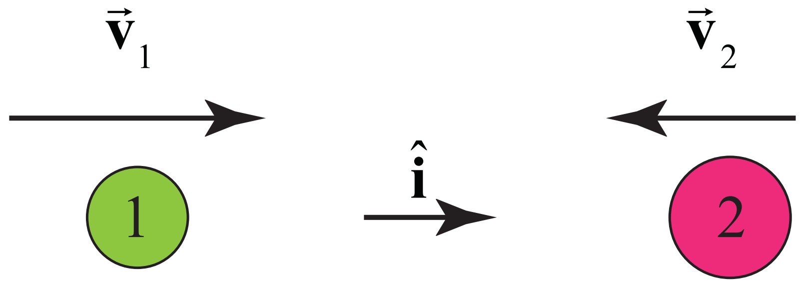
Group Prob. One Dim. Elastic Collision



Consider the elastic collision of two carts on a frictionless track; the incident cart 1 has mass m_1 and moves with initial speed $v_{1,i}$. The target cart 2 has mass $m_2 = 2\ m_1$ and is initially at rest. Immediately after the collision, the incident cart has final speed $v_{1,f}$ and the target cart has final speed $v_{2,f}$.

Find the final velocities of the carts as a function of the initial speed $v_{1,i}$.

Relative Velocity



reference frame fixed to ground

In a reference frame fixed to the ground, two objects are moving with velocities

$$\vec{\mathbf{v}}_1 = v_{1,x}\hat{\mathbf{i}} \quad \text{and} \quad \vec{\mathbf{v}}_2 = v_{2,x}\hat{\mathbf{i}}$$

The relative velocity of object 1 with respect to object 2 is defined to be

$$\vec{\mathbf{v}}^{rel} \equiv \vec{\mathbf{v}}_{2,1} \equiv \vec{\mathbf{v}}_1 - \vec{\mathbf{v}}_2 = (v_{1,x} - v_{2,x})\hat{\mathbf{i}}$$

1-Dim Elastic Collision: Combined Momentum and Energy Principles

Momentum Condition (not restricted to elastic collision):

$$m_1(v_{1,x,i} - v_{1,x,f}) = m_2(v_{2,x,f} - v_{2,x,i}) \qquad (1)$$

Energy Condition (Only holds for 1-Dim. elastic collisions)

$$m_1(v_{1,x,i} + v_{1,x,f})(v_{1,x,i} - v_{1,x,f}) = m_2(v_{2,x,f} + v_{2,x,i})(v_{2,x,f} - v_{2,x,i}) \qquad (2)$$

Divide Eq. (2) by Eq. (1): $v_{1,x,i} + v_{1,x,f} = v_{2,x,f} + v_{2,x,i} \Rightarrow$

$$\boxed{v_{1,x,i} - v_{2,x,i} = v_{2,x,f} - v_{1,x,f}} \qquad (3)$$

Eqs. (1) and (3) are the combined energy and momentum conditions but now in a linear form (no quadratic terms)

Relative Velocity: 1-dim. Elastic Collision

Energy/Momentum Law $$\boxed{v_{1,x,i} - v_{2,x,i} = v_{2,x,f} - v_{1,x,f}} \qquad (3)$$

Components of relative velocities:

$$v_{x,i}^{rel} \equiv v_{1,x,i} - v_{2,x,i}$$

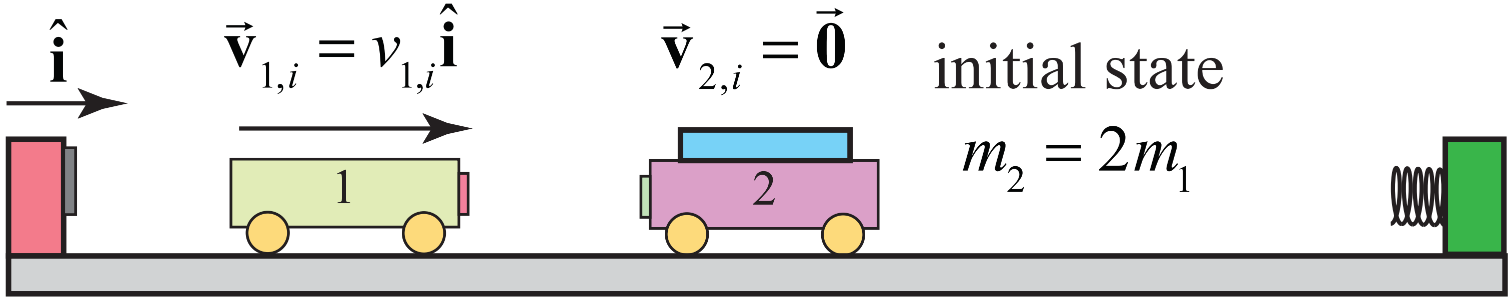
$$v_{x,f}^{rel} \equiv v_{1,x,f} - v_{2,x,f}$$

Relative velocity stays constant in magnitude but direction changes by 180 degrees

$$v_{x,i}^{rel} = -v_{x,f}^{rel}$$

$$\boxed{\vec{\mathbf{v}}_i^{rel} = -\vec{\mathbf{v}}_f^{rel}}$$

Group Prob. One Dim. Elastic Collision Using Relative Velocity Law



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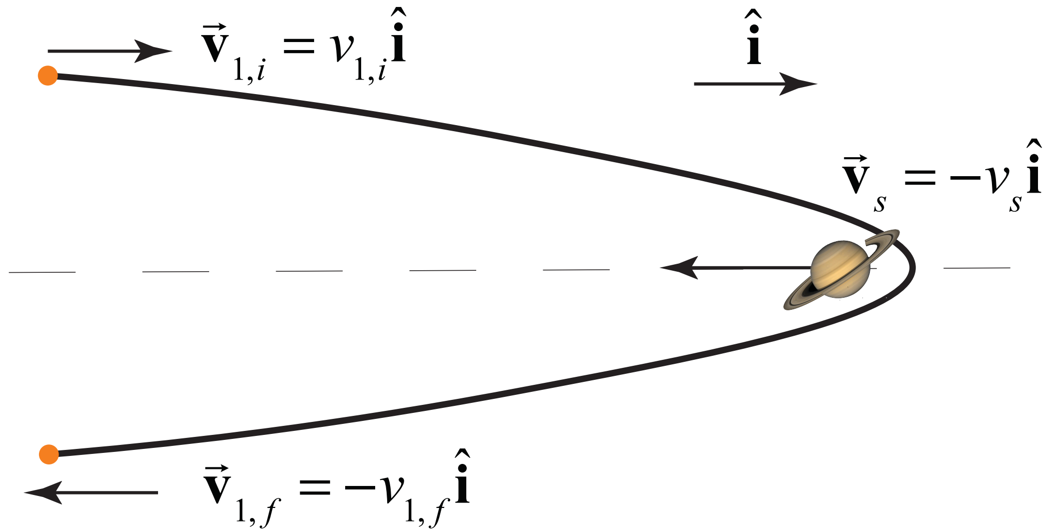
Now use relative velocity and momentum laws to find the final velocities of the carts as a function of the initial speed $v_{1,i}$.

$$v_{1,x,i} - v_{2,x,i} = v_{2,x,f} - v_{1,x,f} \qquad \text{relative velocity law}$$

$$m_1 v_{1,x,i} + m_2 v_{2,x,i} = m_1 v_{1,x,f} + m_2 v_{2,x,f} \quad \text{momentum law}$$

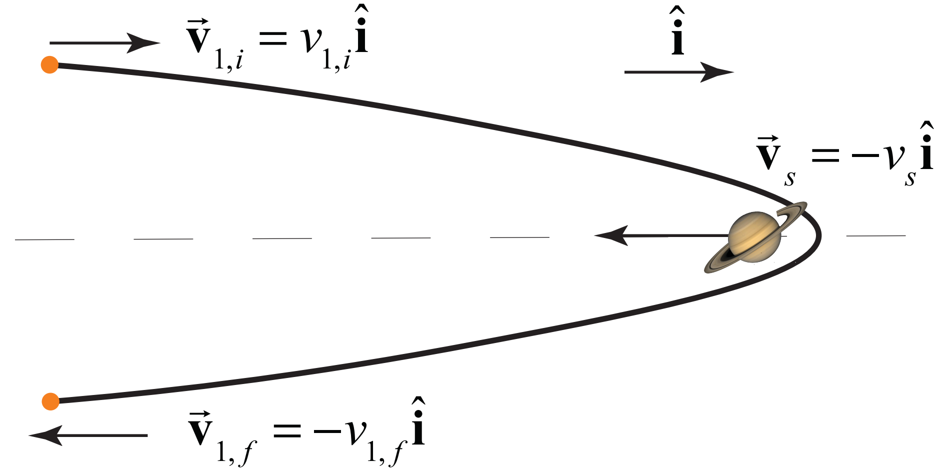
Worked Ex. Gravitational Slingshot



A spacecraft of mass m_1 with speed $v_{1i} = 3v_s$, approaches Saturn which is moving in the opposite direction with speed v_s. After interacting gravitationally with Saturn, the spacecraft swings around Saturn and heads off in the opposite direction from its approach direction. The mass of Saturn is $m_s >> m_1$.

Find the final speed, v_{1f} , of the spacecraft after it is far enough away from Saturn to be nearly free of Saturn's gravitational pull.

Worked Ex. Saturn Fly-by Solution



Change in Saturn's speed is negligible.

$$\vec{\mathbf{v}}_s = \vec{\mathbf{v}}_{s,i} = \vec{\mathbf{v}}_{s,f}$$

Relative Velocities:

$$\vec{\mathbf{v}}_i^{rel} = \vec{\mathbf{v}}_{1,i} - \vec{\mathbf{v}}_{s,i} = (3v_s\hat{\mathbf{i}} - (-v_s\hat{\mathbf{i}}) = 4v_s\,\hat{\mathbf{i}}$$

$$\vec{\mathbf{v}}_f^{rel} = \vec{\mathbf{v}}_{1,f} - \vec{\mathbf{v}}_{s,f} - (v_{1,f}\,\hat{\mathbf{i}}) - (-v_s\,\hat{\mathbf{i}}) = (-v_f + v_s)\hat{\mathbf{i}}$$

Energy/Momentum Law:

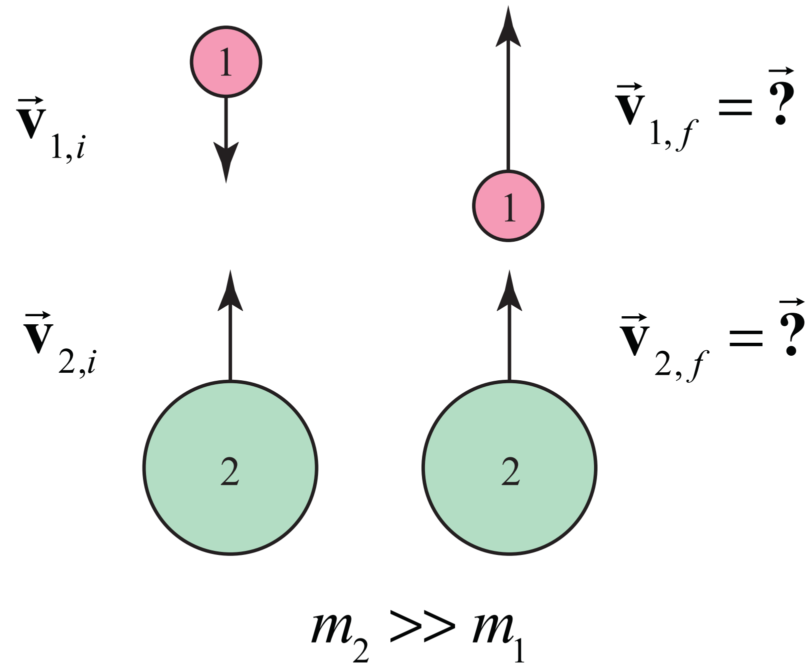
$$\vec{\mathbf{v}}_i^{rel} = -\vec{\mathbf{v}}_f^{rel} \Rightarrow 4v_s\,\hat{\mathbf{i}} = -(-v_f + v_s)\hat{\mathbf{i}}$$

Solve for final speed of satellite:

$$v_f = 5v_s$$

CQ Elastic Collision off a Moving Heavy Object

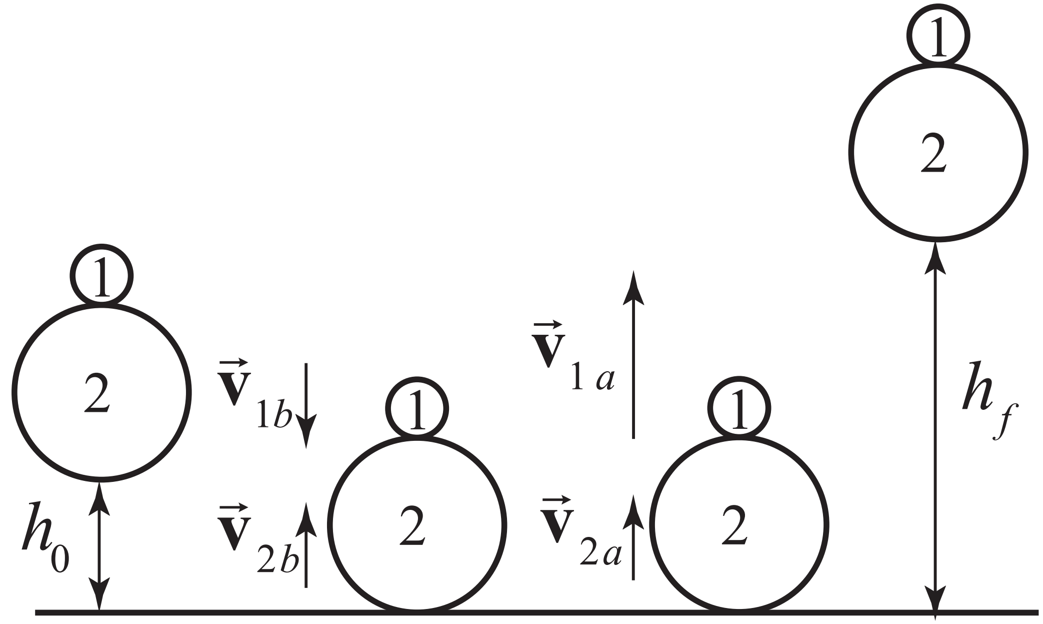
A light object collides elastically with a much heavier object in one dimension that is moving up with the speed v_0. If the heavy object is initially moving, what is the approximate final velocity of the light object?



1. $\vec{\mathbf{v}}_{1f} = -\vec{\mathbf{v}}_{1i} + \vec{\mathbf{v}}_{2i}$
2. $\vec{\mathbf{v}}_{1f} = -\vec{\mathbf{v}}_{1i} - \vec{\mathbf{v}}_{2i}$
3. $\vec{\mathbf{v}}_{1f} = -\vec{\mathbf{v}}_{1i} + 2\vec{\mathbf{v}}_{2i}$
4. $\vec{\mathbf{v}}_{1f} = -\vec{\mathbf{v}}_{1i} - 2\vec{\mathbf{v}}_{2i}$

Group Problem: Two Ball Bounce

Two superballs are dropped from a height h_0 above the ground. Ball 1 has a mass m_1. Ball 2 underneath has a mass m_2, where $m_2 >> m_1$.



Assume that the ball 2 collides elastically with the ground. Then as ball 2 starts to move upward, it immediately collides elastically with ball 1 that is still moving downwards. **Use Relative Velocity Law to find**

a) Find $v_{1,y,a}$, the speed of ball 1 immediately after colliding with ball 2.

b) Find h_f, the height that the ball will rise to?

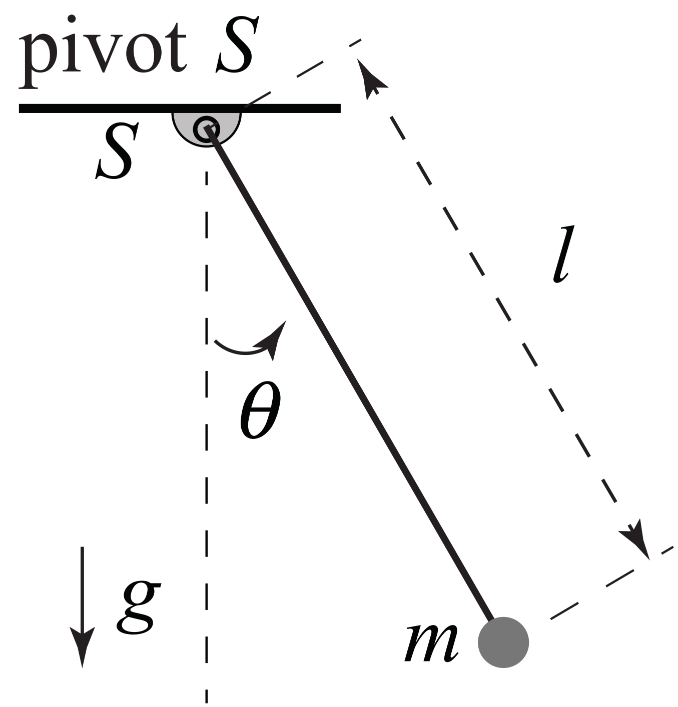
Appendix

Demo: Mini-Experiment: Astro-Blaster

Demo
Simple Pendulum

Group Problem: Simple Pendulum by the Energy Method

a) Find an expression for the mechanical energy $E = K+U$ when the pendulum is in motion in terms of $\theta(t)$ and its first and second derivatives, m, l, and g as needed. Recall the $v = l\, d\theta/dt$.



b) Find the equation of motion for $\theta(t)$ by setting $dE/dt = 0$.

c) Assume that $\theta(t)$ is small i.e. $\sin\theta(t) \cong \theta(t)$, $\cos\theta(t) \cong 1-(1/2)\,\theta(t)^2$.

d) Find ω_0

Periodic vs. Harmonic

Equation of motion:

$$-\frac{g}{l}\sin\theta = \frac{d^2\theta}{dt^2} \Rightarrow periodic$$

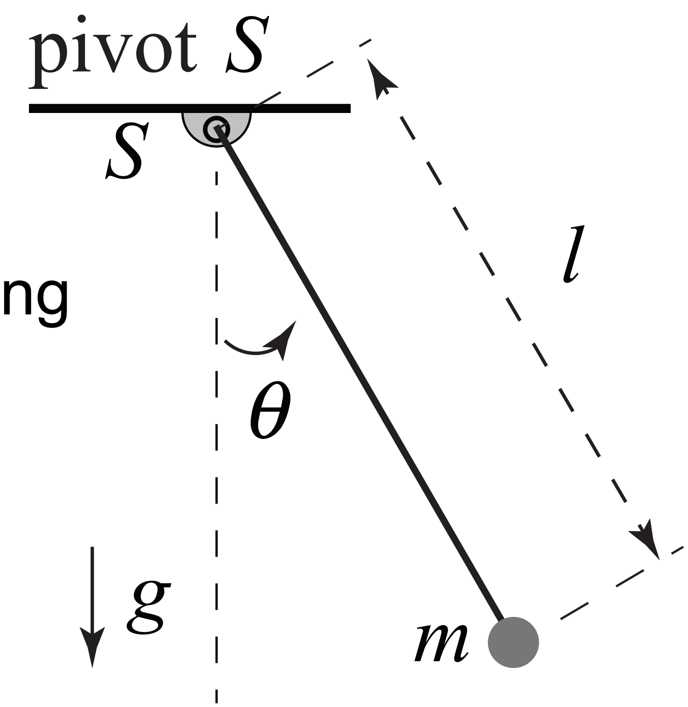
Angle of oscillation is small, linear restoring torque:

$$\sin\theta \cong \theta$$

Simple harmonic oscillator:

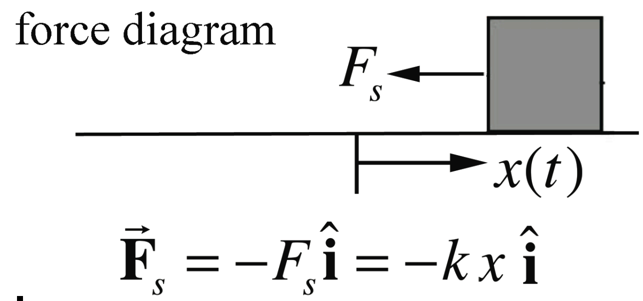
$$\frac{d^2\theta}{dt^2} \cong -\frac{g}{l}\theta \Rightarrow SHO$$

Angular frequency for SHM is independent of amplitude: $\omega_0 \cong \sqrt{g / l}$



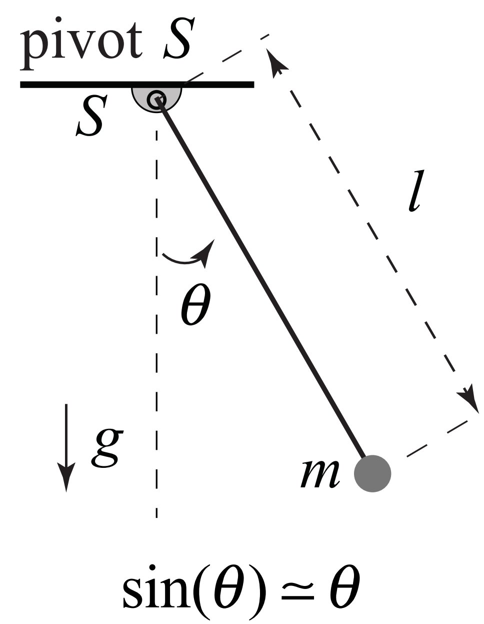
Summary: SHO

Systems with linear restoring forces



1. Ideal spring with $F_x = -kx$

2. Simple pendulum undergoing small angle oscillations



In order to find the equation of motion for an ideal spring we will use two different approaches:

Dynamical Approach: Using Newton's Second Law

$$-kx = md^2x/dt^2$$

Energy Approach: Setting

$$dE/dt = \frac{d}{dt}(K+U) = 0$$