MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Physics

Physics 8.01x Fall Term 2001

EXAM 1 SOLUTIONS

Problem 1:

We define a vertical coordinate system with positive upwards. The only forces acting on the masses are gravity and the tension in the massless rope.

a.) Using Newton's second law in the vertical direction, we can write down the equations of motion for each mass:

$$\sum F_1 = m_1 a_1$$
$$-m_1 g + T = m_1 a_1$$
$$\sum F_2 = m_2 a_2$$
$$-m_2 g + T = m_2 a_2$$

We note that the length of the rope is constant, so an acceleration on one side must be matched by an opposite acceleration on the other side. Therefore we can define a single acceleration a that describes the motion of both masses:

$$a = a_1 = -a_2$$

Subtracting the two equations of motion to cancel out the dependence on the unknown tension gives us:

$$-m_1 g + m_2 g = m_1 a + m_2 a$$
$$a = g\frac{m_2 - m_1}{m_1 + m_2}$$

which agrees with what we might expect: if the first mass is larger than the second, its acceleration will be downward (negative). If you chose the opposite direction for your coordinate system, or chose to solve for the acceleration of the second mass instead of the first, your answer would have the opposite sign as this one.

b.) Now knowing the acceleration, we can plug it back into one of our original force equations and solve for T:

$$T = m_1 a + m_1 g$$
$$T = m_1\left(g\frac{m_2 - m_1}{m_1 + m_2}\right) + m_1 g$$
$$T = \frac{2 m_1 m_2 g}{m_1 + m_2}$$

This also agrees with our intuition: if the two masses are equal, we have a static case, and the tension should just equal to the weight of each mass. In addition, we can note that the tension depends symmetrically on the value of each mass, as we would expect.

c.) This part of the problem introduces kinematic elements. First we should solve for the acceleration of the Atwood machine, and then we can use our result from part (a) to recover the value of g given the masses and acceleration. With basic kinematic equations, we can rearrange terms to be able to solve for acceleration given a final velocity and a final distance:

$$x = \frac{1}{2}at^2$$
$$v = at$$
$$x = \frac{1}{2}a\left(\frac{v}{a}\right)^2$$
$$a = \frac{v^2}{2x}$$

Plugging in our numbers gives the acceleration of the Atwood machine:

$$a = \frac{(-1.5\text{ m/s})^2}{2(-1\text{ m})} = -1.13\text{ m/s}^2$$

Now we use the masses in our formula from part (a). Note that in the above calculation we have found the acceleration of the second, heavier, mass rather than the first, so a little sign flip is necessary to make the connection with part (a):

$$1.13\text{ m/s}^2 = g\frac{0.525\text{ kg} - 0.5\text{ kg}}{0.5\text{ kg} + 0.525\text{ kg}}$$
$$g = 46.1\text{ m/s}^2$$

While our intuition can't verify the actual value, since the planet Mongo is unknown, the dimensions of the answer are correct, and the magnitude is at least in the ballpark of planetary surface accelerations.

Problem 2:

Here we need to reconstruct the initial velocity of Dudley's swing, based on its range. Then we can use that same value to calculate the range in the lunar case. A little bit of quick kinematics derives our equation for the range of a projectile on a level surface, if you did not happen to have it memorized already:

$$v_y(t) = v_0 \sin\theta - g\,t$$

$$t_{total} = \frac{2v_0 \sin\theta}{g}$$

$$x(t) = v_x\,t = v_0\,t\cos\theta$$

$$D = v_0\left(\frac{2\,v_0 \sin\theta}{g}\right)\cos\theta = \frac{v_0^2 \sin(2\theta)}{g}$$

where we have used the fact that on impact, the vertical velocity is the opposite of the initial vertical velocity (that gives the factor of 2 in the second equation). The handy trigonometric identity sin (2θ) = 2 sinθ cos θ was also used. Plugging in our values gives:

$$100\text{ m} = \frac{v_0^2 \sin 60^\circ}{10\text{ m/s}^2}$$

$$v_0 = 34.0\text{ m/s}$$

which is a reasonable magnitude for a human golf swing (76 mph).

a.) We set up a standard coordinate system with the origin at the ball's initial position, and positive directions upwards and in the path of its travel. To find the time, we only need to consider the vertical motion of the ball. It is initially launched upwards at v_0 sin θ, and finally lands at y = –20 m. Note that now the gravitational acceleration we should use is one-sixth of the Earth's: we will continue to use g for Earth's surface gravity.

$$y(t) = v_0\,t\sin\theta - \frac{1}{2}\left(\frac{g}{6}\right)t^2$$

$$-20\text{ m} = (34.0\text{ m/s})\;t\sin 30^\circ - \frac{1}{12}(10\text{ m/s}^2)\,t^2$$

Using the quadratic formula, we can solve for t (leaving off the units for clarity):

$$t = \frac{17.0 \pm \sqrt{17.0^2 + 4\cdot 0.833\cdot 20.0}}{2\cdot 0.833}$$

$$t = 21.5\text{ sec} \quad \text{or} \quad t = -1.12\text{ sec}$$

Only the positive answer is relevant to this problem's situation.

b.) The horizontal distance is straightforward once we know the time, since the horizontal velocity remains constant throughout the flight.

$$x(t) = v_0 \, t \cos\theta$$

$$x_f = (34.0 \text{ m/s})(21.5 \text{ sec})\cos 30^{\circ} = 633 \text{ m}$$

c.) To solve this problem, we could take the approach of parts (a) and (b) and re-solve for the time of flight on a level surface (you would get t = 20.4 sec). A faster approach, however, is to take note of how our original range equation depends on the gravitational acceleration.

The range, as we can see, is inversely proportional to g, so when all other factors are equal (as they are here), if the gravity is six times less, the range will be six times longer:

$$x_f = 6 \cdot D = 600 \text{ m}$$

This compares well to our answer for part (b), since we expect the range to be a little shorter when the ball doesn't have the extra 20 m vertical drop to traverse at the end.

Problem 3:

There are several approaches to extracting the time constant from the data, with tradeoffs between ease and reliability.

The easiest approach is to make use of the fact that an exponential curve is approximately linear close to the origin. Using this, we can extract the time constant from the initial slope obtained from the first two data points.

$$h = h_0 e^{-t/\tau} \approx h_0(1 - t/\tau)$$
$$h \approx h_0 - \frac{h_0}{\tau} t$$

Since we are starting with an approximation anyway, taking h_0 = 8 cm instead of a slightly larger value to accomodate the extra 0.1 sec will not be a major error (especially since it is unlikely that the rest of the time values are accurate to 0.1 sec anyway). Therefore we can get τ directly from the slope of the first two points:

$$-\frac{8\,\text{cm}}{\tau} = \text{slope} = \frac{7\text{ cm} - 8\text{ cm}}{7\,\text{sec} - 0.1\,\text{sec}}$$
$$\tau = 55\,\text{sec}$$

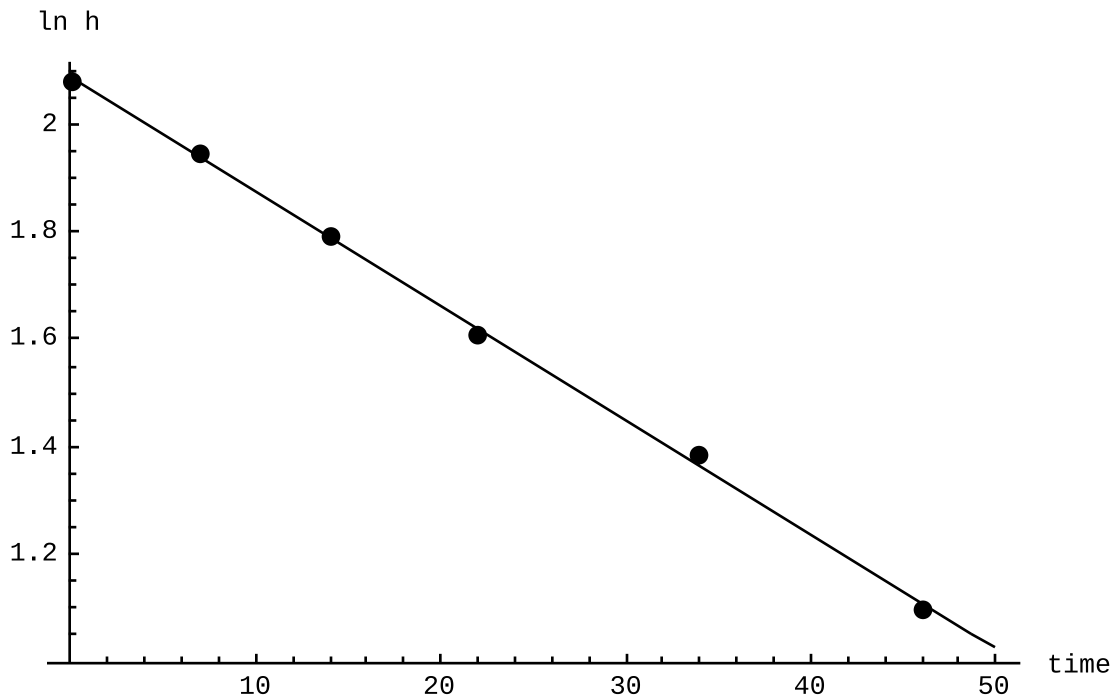
The second approach is to try to identify the 1/*e* point on the time curve, which will fix the time constant. Taking again h_0 = 8 cm, we want to find the time at which the head has dropped to 8 cm (1/*e*) = 2.94 cm. This is just a little bit past our last data point, but we can extrapolate from the last two data points quite easily:

$$\tau = 46\,\text{sec} + (3.0\text{ cm} - 2.94\,\text{cm})\frac{46\,\text{sec} - 34\text{ sec}}{4\text{ cm} - 3\,\text{cm}}$$
$$\tau = 47\,\text{sec}$$

The third approach, which is most reliable, makes use of all the data together, instead of just relying on the first or the last couple of points. Here we plot the data on a semi-log graph, and extract the exponential time constant from the slope of the best-fit line.

Taking the log of the equation turns the exponential into a linear function, and as we can see, the time constant comes directly from the slope of that line:

$$h = h_0 e^{-t/\tau}$$
$$\ln h = \ln h_0 - \frac{1}{\tau} t$$



Here the best-fit line has a slope of -0.0213, which means that:

$$-\frac{1}{\tau} = -0.213$$
$$\tau = 47\,\text{sec}$$

This should be a very reliable answer. In lieu of a formal function fitting (which would have to be done by hand on the test), taking the log slope between the first and last points should serve nearly as well:

$$-\frac{1}{\tau} = \frac{\ln 3 - \ln 8}{46\,\text{sec} - 0.1\,\text{sec}}$$
$$\tau = 47\,\text{sec}$$

As can be seen, all the approaches give approximately the same answer, and any one of them could safely be used on the test.