Clarification for Wednesday November 12

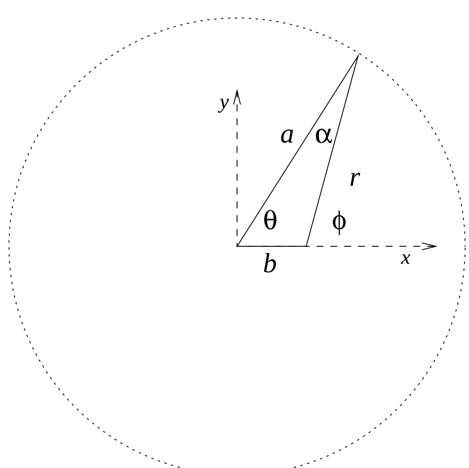
In transcribing my notes regarding the magnetic field due a uniform current in a conducting tube, I may have done things inefficiently or (try not to act too surprised here) incorrectly. If your notes have something different from what follows, you probably copied my mistakes, but the original mistakes have gone the way of all chalk.

So, for the purposed of clarity, completeness and neatness, as well as a more consistent notation, here's what I tried to do on Wednesday, but in better hand-writing.

In the figure below, we have a depiction of the situation considered; a conducting tube, coaxial with the z-axis, with circular cross-section of radius a, carries a uniform current I, directed out of the plane of the paper (that would be the z-direction). We then have a surface current density, that is, current per unit circumferential length, of magnitude $K = I/(2\pi a)$. We wish to find the magnetic field at a point on the x-axis a distance b from the origin.

We found from symmetry considerations that there can be no magnetic field in the z or x-directions. We also found that the contribution to the y-component of the magnetic field due to the current in the tube subtended by the differential angle $d\theta$ (not shown in the diagram) is

$$dB_y = -\frac{\mu_0}{2\pi}\,(K\,a\,d\theta)\,\frac{\cos\phi}{r} = -\frac{\mu_0\,K\,a}{2\pi}\,\frac{\cos\phi\,d\theta}{r},$$



where the angles and the distance r are defined by the figure. Note that r is *not* the radius of the circle, but the distance from the current element $dI = K\,a\,d\theta$ to the point on the x-axis where we want to find the field. Note also that for θ and hence ϕ in the first quadrant, dB_y is negative, as expected. Note even further that I can't get the graphics program to give me ϕ in the font I want.

The fact is, we have too darn many variables floating around. As we change θ to include the entire circle, ϕ, α and r change as well. We can use the easy relation $\phi = \theta + \alpha$, but in hindsight this is not the best thing. This is where I tried to be too tricky. Here's the best way (judgement call, there).

From basic geometry, we have

$$\begin{aligned} r\sin\alpha - b\,\sin\theta &= 0 \\ r\,\cos\alpha + b\,\cos\theta &= b \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} dr\,\sin\alpha + r\,d\alpha\cos\alpha - b\,d\theta\,\cos\theta &= 0 \\ dr\,\cos\alpha - r\,d\alpha\,\sin\alpha - b\,d\theta\,\sin\theta &= 0. \end{aligned}$$

The huge advantage to this is that we can choose which variable we want to use. Knowing which will work best is often a matter of hindsight. For instance, I'm adapting this from *Newtonian Mechanics* by A. P. French, Pages 446-448, which is really neat if you'd like to see a related argument for gravitational fields. The thing is, here we have essentially a two-dimensional problem.

Anyway, the form of dB_y suggests an angular integral, so eliminating dr in the set of equations on the right above, by multiplying the first by $\cos\alpha$ and the second by $\sin\alpha$ and subtracting gives

$$r\,d\alpha - b\,d\theta\,(\cos\theta\cos\alpha - \sin\theta\sin\alpha) = r\,d\alpha - b\,d\theta\,\cos\phi = 0,$$

which then gives

$$dB_y = -\frac{\mu_0\,K\,a}{2\pi}\,\frac{d\alpha}{b}.$$

Now, here's the kicker, and my handwritten notes show a mistake, one which might have propagated onto the board on Wednesday. As we go around the circle, α increases from 0 to $\pi/2$, decreases from $\pi/2$ to 0, then becomes negative, decreasing to $-\pi/2$ and then increasing back to 0. Thus, for a point inside the cylinder, $B_y = 0$ and hence by symmetry $\vec{\boldsymbol{B}} = \vec{\boldsymbol{0}}$.

Another way to see this is to note that since $\phi = \theta + \alpha$, $\Delta\phi = \Delta\theta + \Delta\alpha$, and in going around the circle, $\Delta\theta = \Delta\alpha = 2\pi$, so $\Delta\alpha = 0$.

But, there's more. In the above, we never had to assume anything about the relation between $|a|$ and $|b|$. What if $|b| > |a|$? As θ varies from 0 to 2π, ϕ increases, attains a maximum value, decreases back to 0, becomes negative but returns to 0; $\Delta\phi = 0$, and so $\Delta\alpha = -\Delta\theta = -2\pi$ with the result

$$B_y = \frac{\mu_0\, K\, a}{2\pi}\,\frac{2\pi}{b} = \frac{\mu_0\, K\, a}{b} = \frac{\mu_0\, I}{2\pi\, b},$$

the same as for the field due to a filamentary current a distance b from the wire.

But, there's still more. Suppose $|a| = |b|$; then (convince yourself!) $\Delta\phi = \pi$, $\Delta\alpha = -\pi$, and $B_y = \mu_0\, I/\,(4\pi\, b)$, the average of the values just inside and just outside the tube.

Could there ppossibly be more? Well, if you insist. Recall that the above form for dB_y in terms of $d\theta$, $\cos\phi$ and r came from

$$\frac{-x\, d\theta}{r^2} = \frac{b - a\cos\theta}{a^2 + b^2 - 2\, ab\, \cos\theta}\, d\theta.$$

Why in the world didn't we just leave it in terms of θ and do that integral? Funny you should ask. Here goes one way; I'm always looking for others, such as what I just did.

Rewrite the numerator as

$$b - a\, \cos\theta = \frac{\left(a^2 + b^2 - 2\, ab\, \cos\theta\right) - a^2 - b^2}{2\, b}.$$

Then,

$$\begin{aligned}\frac{b - a\cos\theta}{a^2 + b^2 - 2\, ab\, \cos\theta}\, d\theta &= \left[\frac{1}{2\, b} + \frac{b^2 - a^2}{2\, b}\,\frac{1}{a^2 + b^2 - 2\, ab\, \cos\theta}\right] d\theta \\ &= \frac{1}{2\, b}\left[1 + \left(b^2 - a^2\right)\,\frac{1}{a^2 + b^2 - 2\, ab\, \cos\theta}\right] d\theta.\end{aligned}$$

The first integral is readily done. The second takes some doing, and a derivation will not be given here. The needed definite integral is

$$\int_0^{2\pi} \frac{d\theta}{u + v\, \cos\theta} = \frac{2\pi}{\sqrt{u^2 - v^2}}, \qquad |u| > |v|.$$

For this case, $u = a^2 + b^2$, $v = -2\, ab$ (you should show that $|u| > |v|$), $u^2 - v^2 = \left(a^2 - b^2\right)^2$, leading to

$$B_y = \frac{\mu_0\, K\, a}{2\pi}\,\frac{2\pi}{2\, b}\left[1 + \frac{b^2 - a^2}{|b^2 - a^2|}\right].$$

Note that we couldn't just say $\sqrt{(a^2 - b^2)} = a^2 - b^2$; we must take the postive root. So, if $a > b > 0$, the term in square brackets is zero. If $b > a > 0$, the term in square brackets is 2, leading to the previous result, $B_y = \mu_0 I/2\pi\, b$. If $b = a$,

$$\frac{b - a\cos\theta}{a^2 + b^2 - 2\, ab\, \cos\theta} = \frac{1}{2\, b},$$

leading to a simple integral and the same result found previously.