Supplemental Notes

Today (Monday, October 16), we didn't quite get to show that the field lines for two infinite oppositely-charged paraxial cylinders are circles. The way to do this is to write the equation for the equipotentials so that the equipotentials are level sets (surfaces if we consider the z-direction, contours if we restrict ourselves to the plane. In any event, the field lines will be circles.)

So, define a function $g(x,\, y)$ by

$$g(x,\, y) = k^2 = \frac{r_1^2}{r_2^2} = \frac{(x-a)^2 + y^2}{(x+a)^2 + y^2}.$$

We have seen that the equipotentials are circles centered on the x-axis at $x = \pm a\frac{k^2+1}{k^2-1}$ and radius $\frac{2a}{k^2-1}$.

Anyway, you can show (okay, it took me a while) that

$$\frac{\partial g}{\partial x} = 4a\frac{(x-a)^2 - y^2}{r_2^4}, \quad \frac{\partial g}{\partial y} = 8a\frac{xy}{r_2^4}.$$

We now wish to express the curve of the field lines by y in terms of x. (We could look for some $y(x)$ explicitly, or y as an implicit function of x. In this case, it turns out not to matter.) The relation we need is from physics; the field lines will be tangent to the gradient of g (note that $g(x,\, y)$ in *not* the potential, but the gradient of g is still tangent to the field lines). Thus,

$$\frac{dy}{dx} = \frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial x}} = \frac{2xy}{x^2 - a^2 - y^2}.$$

There are many ways to solve such an item; the way presented below is one of the most direct, but it does use some knowledge of 18.03 and a bit of 18.02 that may not have been presented. That is, we rewrite the above as

$$2xy\, dx - (x^2 - a^2 - y^2)\, dy = 0,$$

and see if (remember, that's an *if* at this stage) this can be rewritten as

$$df = \frac{\partial f}{\partial x}\, dx + \frac{\partial f}{\partial y}\, dy = 0.$$

Well, it can't. If it could, the term $2xy$ would be $\frac{\partial f}{\partial x}$ so that $\frac{\partial^2 f}{\partial y \partial x} = 2x$, whereas the term $-(x^2 - a^2 - y^2)$ would be $\frac{\partial f}{\partial y}$ so that $\frac{\partial^2 f}{\partial x \partial y} = -2x$; close but no cigar.

So, what we do is look for an integrating factor; the details of how to do this take a while and don't add much to the physics (and they're hard to type), so I'll just say that we mulitply by a factor of y^{-2}, and this works, in that we now have

$$\frac{2x}{y}\,dx - \left(\frac{x^2}{y^2} - \frac{a^2}{y^2} - 1\right)\,dy = 0.$$

Fill in the details yourself, but the result is

$$\begin{aligned} \frac{2x}{y} &= \frac{\partial}{\partial x}\left(\frac{x^2}{y} - \frac{a^2}{y} + y\right) \\ -\left(\frac{x^2}{y^2} - \frac{a^2}{y^2} - 1\right) &= \frac{\partial}{\partial y}\left(\frac{x^2}{y} - \frac{a^2}{y} + y\right), \end{aligned}$$

and so the field lines follow a curve given by

$$\left(\frac{x^2}{y} - \frac{a^2}{y} + y\right) = c,$$

where c is a constant of integration. Rearranging gives

$$x^2 + (y - c/2)^2 = a^2 + c^2/4.$$

In this form, remember that c could be positive or negative; this might help you decide what the circles look like.